SCHUR FUNCTIONS AND DOMINO TABLEAUX
GENERALISATION TO K-THEORY

FLORENCE MAAS-GARIÉPY

1. Introduction 2
2. Symmetric functions, partitions and Young tableaux 2
2.1. Symmetric functions 2
2.2. Partitions and Young tableaux 4
2.3. Best known bases of the set of symmetric functions, Λ 7
3. Schur functions 11
4. Multiplication of the Schur functions 13
4.1. 2-quotient and 2-core 13
4.2. Domino tableaux 14
4.3. Bijection 16
5. K-theory 18
5.1. Stable Grothendieck polynomial G_λ 18
5.2. Multiplication of stable Grothendieck polynomials 20
6. Word insertion through the RSK algorithm, Hecke insertion and unique rectification target 22
6.1. RSK algorithm and Knuth equivalence classes 22
6.2. Hecke insertion and K-Knuth equivalence classes 24
6.3. K-Knuth equivalence classes 26
7. Colored permutation insertion into a domino tableau and unique rectification domino tableau 27
7.1. Domino insertion 27
7.2. Domino "Hecke" insertion, domino insertion classes and unique rectification domino 32
8. Conclusion 34
Remerciements 34
References 35

Contents

Date: August 6, 2017.

1. Introduction

Symmetric functions are very often studied in combinatorics, especially the symmetric functions called Schur functions. They have connections to the representation theory of certain groups, and allows to describe naturally varieties over certain vector spaces. We will focus in this articles on the Schur functions, their multiplication and their K-theoric generalisation to the Grothendieck polynomials.

We also describe Knuth equivalence classes and K-Knuth equivalence classes and some result of generalising them to the notion of domino tableaux.

The combinatoric point of view of the symmetric functions is based on important combinatoric objects that we will describe here: partitions and Young tableaux. No previous knowledge in symmetric functions or combinatorics should be needed to read this article.

2. Symmetric functions, partitions and Young tableaux

We will start by making some general reminders on symmetric functions, partitions, Young tableaux and Schur functions.

A reader comfortable with these notions may want to start directly at the section 3, 4 or 5. The next four sections are based on the sections 7.1 to 7.12 in [1], we refer the reader to this excellent reference for more details.

2.1. Symmetric functions.

Definition 2.1. Let $x = (x_1, x_2, x_3, ..., x_n)$, be a set of variables, for $n \in \mathbb{N}$.

A **symmetric function** is a polynomial or a formal power series

$$f(x) = \sum_{\alpha} c_\alpha x^\alpha,$$

where

- $c_\alpha \in R$, for R a commutative ring with an identity.
- $\alpha = (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n)$, for $\alpha_i \in \mathbb{N}$.
- $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} ... x_n^{\alpha_n}$
- $\forall \sigma \in \S_n$, the symmetric group, $f(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}) = f(x_1, x_2, ..., x_n)$.

Example 2.2. $f(x, y, z) = x + y + z + x^2 y + y^2 x + x^2 z + y^2 z + z^2 x + z^2 y + x^3 y^3 z^3$ is a symmetric function.

Counter-example 2.3. $g(x,y) = x + x^2 + y^2 + xy + x^2y^3$ is not a symmetric function. We would need to add the following terms for g to be a symmetric function : y and x^3y^2.

Definition 2.4. A **homogeneous symmetric function of degree n** is a symmetric function with all its monomes having degree n.

In other words, if such a symmetric function is given by $f(x) = \sum_\alpha c_\alpha x^\alpha$, then

$$\forall \alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots), \sum_i \alpha_i = n.$$

Example 2.5. $f(x_1, x_2, x_3, x_4, x_5) = x_1 + x_2 + x_3 + x_4 + x_5$ is a homogeneous symmetric function of degree 1, in 5 variables.

Example 2.6. $g(x,y,z) = x^2y + x^2z + y^2x + y^2z + z^2x + z^2y + xyz$ is a homogeneous symmetric function of degree 3, in 3 variables.

In general, when speaking of symmetric functions, $x$is an **"infinite" set of variables**. It means that $x = (x_1, x_2, x_3, \ldots)$ and $f(x)$ is a formal power series.

We then suppose that $n \in \mathbb{N}$, permuting **any n variables** always gives back $f(x)$.

Example 2.7. Let $x = (x_1, x_2, x_3, \ldots)$, and

$$f(x) = \sum_i 3x_i^4 + \sum_{i \neq j} x_i^2 x_j^5,$$

where $i \in \mathbb{N}$. Then $f(x)$ is a homogeneous symmetric function of degree 5, in an infinity of variables.

We denote the **set of homogeneou symmetric functions of degree n over $R = \mathbb{Q}$** by Λ^n.

Let's note that the homogeneous symmetric functions of degree 0 sont are simply the elements of $R = \mathbb{Q}$, and that if $f \in \Lambda^n$ and $g \in \Lambda^m$, then $f \cdot g \in \Lambda^{n+m}$.

We denote the **set of symmetric functions over $R = \mathbb{Q}$** by Λ. We have that $\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \ldots$.

Indeed, if $f \in \Lambda$, then $f = f_0 + f_1 + f_2 + \ldots$ where $f_i \in \Lambda^n$.

Λ is in fact an algebra with identity $1 \in \Lambda^0$, in other words a ring with operations compatible with the structure of $\mathbb{Q}$-vector space. One of the goals of the study of symmetric functions is to describe "good" basis of Λ.

2.2. **Partitions and Young tableaux.** To describe the best known bases of Λ, we first have to introduce partitions and Young tableaux.

Definition 2.8. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ for $\lambda_i \in \mathbb{N}$. We say that λ is a **partition of n** if

(1) $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_k = n$ and

(2) $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_k > 0$.

We then denote $\lambda \vdash n$, and we say that the **length of λ**is the number of parts of λ, which we denote $\ell(\lambda) = k$.

Example 2.9. (1) $\lambda = (2,1,1)$ is a partition of 4.

(2) $\lambda = (6,5,5,4,2,1,1,1,1)$ is a partition of 26.

(3) $\lambda = (3,4,2,2,1)$ is NOT a partition.

Notation 2.10. The **set of partitions of n** is denoted $Par(n)$, and the **set of partitions** is denoted Par.

Given two partitions, it can be useful to know if one if "greater" than the other. This **dominance order** is defined as follows:

Definition 2.11. Let λ, $\mu \in Par(n)$. We say that $\lambda \leqslant \mu$ if

$\lambda_1 + \lambda_2 + ... + \lambda_i \leqslant \mu_1 + \mu_2 + ... + \mu_i \; \forall i \geqslant 1.$

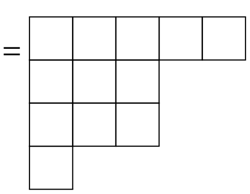
We can also want to know if one of two given partitions is included into the other. This **inclusion order** is defined as follows:

Definition 2.12. Let λ, $\mu \in Par$. We say that $\lambda \subseteq \mu$ if $\lambda_i \leqslant \mu_i \; \forall i \geqslant 1$.

Now that we have introduced partitions, we can move on to describing Young diagrams and Young tableaux.

Definition 2.13. Let λ be a partition of n. A **Young diagram of shape λ** is a left justified empty tableau with n cells ordered such that the i^{th} line contains λ_i cells.

Example 2.14. Let $\lambda = (5,3,3,1)$. The Young diagram of shape λ is

$D_\lambda =$ 

In this article, we use the english notation. In this notation, the lines are written from top to bottom, as in the french notation, lines are written from bottom to top. A reader wishing to study symmetric functions or tableaux will probably encounter both, as they exist simultaneously in different books and articles, depending on the preferences of the authors.

An tableau made of empty cells is a pretty sad mathematical object, we then wish to fill it with integers in the following way.

Definition 2.15. Let λ be a partition of n. A **(semi-standard) Young tableau of shape** $\lambda$is the filling of a Young diagram with positive integers such that:

- Lines are weakly incresing from left to right.
- Columns are strictly increasing form top to bottom.

Example 2.16. Let $\lambda = (4,2,1,1)$.The following tableau is a Young tableau of shape λ.

$T =$

1	2	2	4
2	4		
3			
5			

We will now introduce some vocabulary associated with Young tableaux.

Definition 2.17. Let λ be a partition of n and T be a Young tableau of shape λ.

We denote the **shape of** T by $sh(T) = \lambda$, and we say that the **size of** T is the number of cells in T, denoted by $|T| = |\lambda| = n$.

Definition 2.18. We say that a **Young tableau is strictly increasing** if all its lines are strictly increasing from left to right.

Example 2.19.

1	3	4	5
2	5		

is a strictly increasing Young tableau.

Definition 2.20. We say that a **Young tableau** T **is standard** if its cells are filled by $\{1,2,3,\dots n\}$, where n is the size of T.

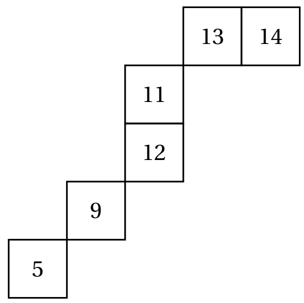
Example 2.21.

1	2	5	6
3	4		

is a standard Young tableau.

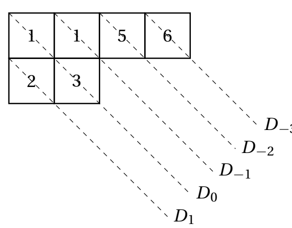
Definition 2.22. Let λ, μ, be two partitions such that $\mu \subseteq \lambda$. The **skew Young diagram of shape λ/μ** is the Young diagramme de Young of shape λ, of which the cells corresponding to the Young diagram of shape μ have been removed. A **skew Young tableau of shape λ/μ** a filling of the skew Young diagram of shape λ/μ such that rows are weakly increasing and columns are strictly increasing. Similarly to Young tableaux, we can describe standard skew tableaux and increasing skew tableaux.

Example 2.23. The following tableau is a skew Young tableau of shape $(5,3,3,2,1)/(3,2,2,1)$.



Definition 2.24. Each cell of a Young tableau is crossed by a unique **diagonal D_k** of equation $D_k = x - k$, for $k \in \mathbb{N}$.

Example 2.25. The following figure represents a Young tableau with its diagonals.



Definition 2.26. Let T be a Young tableau. There are two main ways to read the entries of T:

(1) The **reading word of T, or word of T,** is the sequence of entries obtained by reading the lines of T from left to right, starting by the bottom line and reading the lines upwards.

(2) The **diagonal reading of T** is the sequences of entries read along the diagonals from top to bottom, starting with the leftmost diagonal and with " / " between the readings of different diagonals.

Example 2.27. Let $T = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 & 3 & 4 \\ \hline 2 & 3 & 3 & 4 \\ \cline{1-4} 4 & 5 & 5 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array}$.

The reading word of T is 545523341112334. The diagonal reading of T is 5/4/25/135/13/14/2/3/3/4.

Remark 2.28. These two types of readings allow to find back the original tableau directly from the word associated with the tableau.

We will now put these new notions of partitions and Young tableaux to use, and use them to describe symmetric functions. To do this we need the following notation.

Definition 2.29. Let λ be a partition of n and T a Young tableau of shape λ.

We say that the **type of T** is the vecteur $\alpha(T) = (\alpha_1(T), \alpha_2(T), \ldots)$, where $\alpha_i(T)$ indicates the number of entries of value i in T.

If $x = (x_1, x_2, x_3, \ldots)$, then we denote $x^T = x^{\alpha(T)} = x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} x_3^{\alpha_3(T)} \ldots$

Example 2.30. Let $T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 4 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$. Then

$$\alpha(T) = (1, 3, 0, 2, 0, 0, \ldots), \text{ and}$$

$$x^T = x_1 x_2^3 x_4^2.$$

2.3. **Best known bases of the set of symmetric functions, Λ.** In this section, we will describe the classical bases of the set of symmetric functions, Λ.

2.3.1. *Monomial symmetric functions.*

Definition 2.31. Let λ be a partition. The **monomial symmetric function associated to λ** is defined by

$$m_\lambda := \sum_\alpha x^\alpha$$

where $\alpha = (\alpha_1, \alpha_2, \ldots)$ runs over the set of distinct permutations of the coordinates of $\lambda = (\lambda_1, \lambda_2, \ldots)$.

Example 2.32. If $\lambda = \varnothing = (0, 0, 0, 0, \ldots)$, then $m_\varnothing := x^{(0,0,0,0,\ldots)} = x_1^0 x_2^0 \ldots = 1$.

If $\lambda = (1) = (1, 0, 0, 0, \ldots)$, then $m_1 := \sum_i x^{(0,0,\ldots,0,1,0,\ldots)} = \sum_i x_i$.

If $\lambda = (2, 1) = (2, 1, 0, 0, \ldots)$, then $m_{2,1} := \sum_{i<j} x_i^2 x_j + \sum_{i<j} x_i x_j^2$.

It is relatively easy to prove that monomial symmetric functions form a basis of the set of symmetric functions. We refer the reader to [1] for the details of this proof. However it is not an equivalently easy proof for all bases of Λ. In fact, the strategy for showing that a basis is actually a basis of Λ is generally to find a change of basis between these symmetric functions and the monomial symmetric functions. We refer the reader again to [1] for the complete proofs that the bases that we will describe further on are really bases of Λ.

2.3.2. *Elementary symmetric functions.*

Definition 2.33. Let $n \in \mathbb{N}$. The **elementary symmetric function associated to n** is defined by

$$e_n := m_{1^n} = \sum_{i_1 < i_2 < \ldots < i_n} x_{i_1} x_{i_2} \ldots x_{i_n} = \sum_{\beta} x^{\beta}$$

where β runs over the set of all fillings of the "column" Young diagram of height n.

Definition 2.34. Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$. The **elementary symmetric function associated to λ** is

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots$$

Example 2.35. Let $\lambda = (3,2,1,1) \vdash 7$. Let's restrain ourselves to the case with 3 variables.

We have that $e_3 := m_{1^3} = \sum_{\beta_3} x^{\beta_3}$ where β_3 runs over the set of all fillings of the "column" Young diagram of height 3.

We have a single possible filling with three variables:

1
2
3

So $e_3(x_1, x_2, x_3) = x_1 x_2 x_3$.

We have that $e_2 := m_{1^2} = \sum_{\beta_2} x^{\beta_2}$ where β_2 runs over the set of all fillings of the "column" Young diagram of height 2.

We have two possible fillings with three variables:

1	2
2	3

So $e_2(x_1, x_2, x_3) = x_1x_2 + x_2x_3$.

Finally, we have that $e_1 := m_1 = \sum_{\beta_1} x^{\beta_1}$, where β_1 runs over the set of all fillings of the "column" Young diagram of height 1.

We have three possible fillings with three variables:

1	2	3

So $e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$.

Finally, we have the following equation for $e_\lambda(x_1, x_2, x_3)$.

$$\begin{aligned} e_\lambda(x_1, x_2, x_3) &= e_3(x_1, x_2, x_3)e_2(x_1, x_2, x_3)e_1(x_1, x_2, x_3)e_1(x_1, x_2, x_3) \\ &= (x_1x_2x_3)(x_1x_2 + x_2x_3)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3) \\ &= 3x_1^3x_2^2x_3^2 + 3x_1^2x_2^3x_3^2 + 3x_1^2x_2^2x_3^3 + 2x_1^3x_2^3x_3 + 2x_1x_2^3x_3^3 \\ &+ x_1^2x_2^4x_3 + x_1^4x_2^2x_3 + x_1x_2^2x_3^4 + x_1x_2^4x_3^2 \end{aligned}$$

2.3.3. *Homogeneous symmetric functions.*

Definition 2.36. Let $n \in \mathbb{N}$. The **homogeneous symmetric function associated to n** is defined by

$$h_n := \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n} x_{i_1} x_{i_2} \ldots x_{i_n} = \sum_{\beta} x^\beta$$

where β runs over the set of all fillings of the "line" Young diagram of length n.

Let's note that h_n gives us the sum of all monoms of degree n.

Definition 2.37. Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$. The **homogeneous symmetric function associated to λ** is

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots$$

Example 2.38. Let $\lambda = (3, 2, 1, 1) \vdash 7$. Let's restrain ourselves again to the case with 3 variables.

We have that $h_3 := \sum_{\beta_3} x^{\beta_3}$ where β_3 where β_3 runs over the set of all fillings of the "line" Young diagram of length 3.

There are many possible fillings with three variables:

1 1 1	1 1 2	1 2 2	1 2 3	1 1 3	1 3 3	2 2 2
		2 2 3	2 3 3	3 3 3		

So $h_3(x_1,x_2,x_3) = x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + x_1x_2x_3$.

We have that $h_2 := \sum_{\beta_2} x^{\beta_2}$ where β_2 where β_2 runs over the set of all fillings of the "line" Young diagram of length 2.

There are six possible fillings with three variables:

$$\boxed{1}\boxed{1} \quad \boxed{1}\boxed{2} \quad \boxed{1}\boxed{3} \quad \boxed{2}\boxed{2} \quad \boxed{2}\boxed{3} \quad \boxed{3}\boxed{3}$$

So $h_2(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$.

Finally, we have that $h_1 = \sum_{\beta_1} x^{\beta_1}$, where β_1 runs over the set of all fillings of the "line" Young diagram of length 2.

There are three possible fillings with three variables:

$$\boxed{1} \quad \boxed{2} \quad \boxed{3}$$

So $h_1(x_1,x_2,x_3) := \sum_{i\in\{1,2,3\}} x_i = x_1 + x_2 + x_3$.

Fanally, we have the following equation for $h_\lambda(x_1,x_2,x_3)$

$$\begin{aligned} h_\lambda(x_1,x_2,x_3) &= h_3(x_1,x_2,x_3)h_2(x_1,x_2,x_3)h_1(x_1,x_2,x_3)h_1(x_1,x_2,x_3) \\ &= (x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + x_1x_2x_3) \\ &\cdot (x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3) \cdot (x_1 + x_2 + x_3) \cdot (x_1 + x_2 + x_3) \end{aligned}$$

2.3.4. *Power sum symmetric functions.*

Definition 2.39. Let $n \in \mathbb{N}$. The **power sum symmetric function associated to n** is defined by

$$p_n := m_n = \sum_i x_i^n.$$

Definition 2.40. Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$, the **power sum symmetric function associated to λ** is

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots$$

Example 2.41. Let $\lambda = (3,2,1,1) \vdash 7$. Let's restrain ourselves again to the case with 3 variables.

We have that $p_3(x_1,x_2,x_3) := m_3(x_1,x_2,x_3) = \sum_{i\in\{1,2,3\}} x_i^3$, so $p_3(x_1,x_2,x_3) = x_1^3 + x_2^3 + x_3^3$.

We have that $p_2(x_1,x_2,x_3) := m_2(x_1,x_2,x_3) = \sum_{i\in\{1,2,3\}} x_i^2$, so $p_2(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2$.

We have that $p_1(x_1,x_2,x_3) = \sum_{i\in\{1,2,3\}} x_i$, so $p_1(x_1,x_2,x_3) = x_1 + x_2 + x_3$.

Finally, we have the following equation for $p_\lambda(x_1, x_2, x_3)$.

$$\begin{aligned} p_\lambda(x_1,x_2,x_3) &= p_3(x_1,x_2,x_3)p_2(x_1,x_2,x_3)p_1(x_1,x_2,x_3)p_1(x_1,x_2,x_3) \\ &= (x_1^3+x_2^3+x_3^3)(x_1^2+x_2^2+x_3^2)(x_1+x_2+x_3)(x_1+x_2+x_3) \end{aligned}$$

2.3.5. *Relations between the bases.* One of the interests of studying the bases of Λ is to study the **matrix of change of basis** between two given bases, which allows to write one element of the first basis in term of the elements of the second basis, and vice versa.

In general, it is easier to use the power sum symmetric functions to express the elements of other bases, by using the following two relations.

Proposition 2.42.

$$h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda$$

$$e_n = \sum_{\lambda \vdash n} \varepsilon_\lambda z_\lambda^{-1} p_\lambda$$

where $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \ldots$ *for* $m_i = \#i$ *in* λ*, and* $\varepsilon_\lambda = (-1)^{n-l(\lambda)}$.

We refer the reader to [1] for other relations between bases of the set of symmetric functions Λ.

3. Schur functions

Schur functions form another basis of Λ, and are especcially interesting since they have different definitions in distinct mathematical fields. We refer the reader to [10] for further details.

To start with, they can be entirely be described combinatorically, which is the definition we will be focussing on in this article.

They also have a definition in terms of representation theory, since the Schur functions have strong connections to the representation theory of the symmetric group S_n and of other related groups. In fact, they are the characters of polynomial irreducible representations of the general linear groups.

Similarly, Schur functions have a geometric definition in Schubert theory. This theory is a branch of algebraic geometry that was introduced in the nineteen century by Hermann Schubert. In this theory, a Grassmanian variety is defined to be a variety whose points are subspaces of a given vector space, usually $\mathbb{C}^n$. The Schur functions then represent the Schubert classes in the cohomology ring of the Grassmannians $Gr(k, \mathbb{C}^n)$ of k-planes in $\mathbb{C}^n$.

There are a few other definitions of the Schur functions, but we won't describe them here. Let's now give the combinatorical definition of Schur functions.

Definition 3.1. Let λ a partition, and $x = (x_1, x_2, \dots, x_n)$ for $n \in \mathbb{N}$. The **Schur function associated to λ in x** is given by the following expression.

$$s_\lambda(x) = \sum_T x^T$$

where T runs over the set of Young tableaux of shape λ.

When $x = (x_1, x_2, x_3, \dots)$, we simply denote $s_\lambda(x)$ by s_λ.

Example 3.2. Let's consider the Young tableaux of shape $\lambda = (2,1)$ with largest entry smaller or equal to 3.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Since the largest entry appearing in the tableaux will be 3, then all terms of the Schur function spanning over these tableaux will be composed uniquely of x_1's, x_2's or x_3's, and is then denoted $s_{(2,1)}(x_1, x_2, x_3)$. We have the following equation for $s_{(2,1)}(x_1, x_2, x_3)$.

$$\begin{aligned} s_{(2,1)}(x_1, x_2, x_3) &= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 \\ &= m_{(2,1)}(x_1, x_2, x_3) + 2m_{(1,1,1)}(x_1, x_2, x_3) \end{aligned}$$

In fact, since there can't be more than three variables in any given term in $s_{(2,1)}$, we then have the following equation for $s_{(2,1)}$.

$$s_{(2,1)} = m_{(2,1)} + 2m_{(1,1,1)}$$

Theorem 3.3. *For all partition λ, the Schur function associated to λ, s_λ, is a **symmetric function**.*

In particular, for $|\lambda| = n$,

$$s_\lambda = \sum_\alpha k_{\lambda,\alpha} x^\alpha = \sum_{\mu \vdash n} k_{\lambda,\mu} m_\mu,$$

where $k_{\lambda,\alpha}$ is the number of Young tableaux of shape λ and type α.

Theorem 3.4. *The Schur functions s_λ with $\lambda \in Par(n)$ form a basis of Λ^n, and therefore the set*

$$\{s_\lambda \mid \lambda \in Par\}$$

forms a basis of Λ.

Theorem 3.5 (Cauchy identity)**.**

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x) s_\lambda(y)$$

Proof. The proof of this theorem is based on the RSK algorithm. This algorithm establishes a bijection between the $\mathbb{N}$ -matrices with finite non-zero entries and such that the vector of the sums of entries in each rows, $row(A)$ is α, and the vector of the sum of entries in each column is β, and pairs of Young tableaux (P, Q) such that $type(P) = \alpha$ et $type(Q) = \beta$. We refer the reader to [**?**] for a complete description of the RSK algorithm.

4. Multiplication of the Schur functions

In representation theory and algebraic geometry, it can be useful to have a formula to describe the multiplication of two Schur function, else than the direct multiplication of two power sums.

In particular, for $\mu, \nu \vdash n$, how can we write $s_\mu s_\nu$ in terms of the Schur basis?

The Littlewood-Richardson provides the coefficients of such an expression. In this section, we will describe another combinatorical interpretation of these coefficients when the multiplication involves particular partitions. In order to understand well this theorem, we will first introduce the notions of **2-quotient, 2-core and domino tableaux**. This whole section is based on the background material found in [2].

4.1. **2-quotient and 2-core.**

Definition 4.1. Let $\lambda \vdash n$. The **2-quotient of λ** is a pair of partitions (μ, ν) obtained in the following way:

(1) Let $L := (l_1, l_2, ..., l_{l(\lambda)})$, where $l_i = \lambda_i + l(\lambda) - i$ for $i \in \{1, 2, ..., l(\lambda)\}$.

(2) Let M be obtained from L by replacing succesively from right to left the even numbers by $0, 2, 4, ...$ and uneven numbers by $1, 3, 5, ...$

(3) We substract the even components of L by the even components of M and we divide by to to obtain μ.

(4) We substract the uneven components of L by the uneven components of M and we divide by to to obtain ν.

Example 4.2. Let $\lambda = (4,2,2,1,1,1)$. Here the length of λ is $\ell(\lambda) = 6$. We then have

(1) $L = (4+6-1, 2+6-2, 2+6-3, 1+6-4, 1+6-5, 1+6-6) = (9,6,5,3,2,1)$

(2) $M = (7,2,5,3,0,1)$

(3) $\mu = \frac{1}{2}((6,2)-(2,0)) = \frac{1}{2}(4,2) = (2,1)$

(4) $\nu = \frac{1}{2}((9,5,3,1)-(7,5,3,1)) = \frac{1}{2}(2,0,0,0) = (1,0,0,0)$

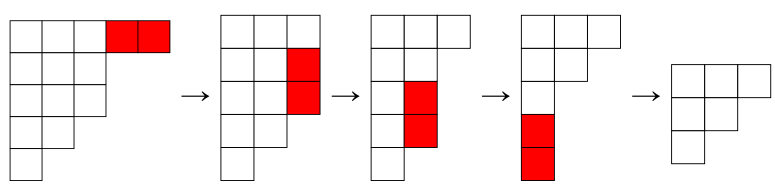
We then have that the 2-quotient of λ is $((2,1),(1))$.

Definition 4.3. Let $\lambda \vdash n$, the **2-core of λ** is obtained by succesively removing dominos (rectangles 2×1 or 1×2) from the Young diagram of shape λ if there are no cells to the right or under that domino.

The Young diagram left when no such domino can be removed is called the 2-core of λ.

Remark 4.4. It has been shown that the 2-core is independant of the order in which the dominos are removed. Also, the 2-core is always of reverse staicase shape. In other words, it has shape $(k, k-1, k-2, \ldots, 2, 1)$ for $k \in \mathbb{N}$.

Example 4.5. Let $\lambda = (5,3,3,2,1,1,1)$. We can obtain the 2-core of λ in the following way:



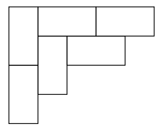
4.2. **Domino tableaux.**

Definition 4.6. If $\lambda \vdash n$ has 2-core $\varnothing$, then it is possible to pave λ with dominos. We then say that λ is **pavable**.

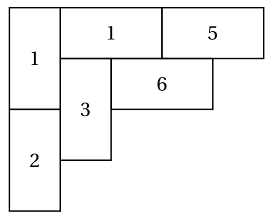
Definition 4.7. A **domino tableau of shape λ** is the filling of a domino paving of λ by positive integers such that:

- Lines are weakly increasing from left to right.
- Columns are strictly increasing from top to bottom.

Example 4.8. Let $\lambda = (5,4,2,1)$. The following figure is a paving of λ.



The following figure is a domino tableau of shape λ and of type $(2,1,1,0,1,1,0,0,0,\dots)$ with such a paving.

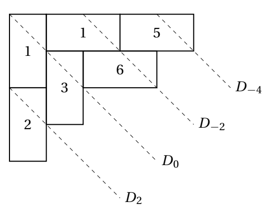


Similarely to Young tableaux, it is possible to read the entries in a domino tableau. Because of the particular positions of the dominos in a tableau, we will privilege the diagonal reading for domino tableaux.

Definition 4.9. Each domino in a domino tableau is crossed by a unique diagonal D_{2k} of equation $D_k = -x - 2k$.

Definition 4.10. We call **diagonal reading** of a domino tableau the integer sequence read along the diagonals from top to bottom, starting by the leftmost diagonal. We separate the entries on distinct diagonals by "/" .

Example 4.11. The following figure represents the domino tableau of the previous example with its diagonals.



The diagonal reading of this tableau is 2 / 1,3 / 1,6 / 5.

4.2.1. *Types of dominos.* We remark that depending on how the diagonal D_{2k} cuts a domino, we can distinguish two types of dominos:

(1) Type 1 : dominos with the small triangle defined by the cutting of the diagonal pointing upwards.

(2) Type 2: dominos with the small triangle defined by the cutting of the diagonal pointing downwards..

4.3. **Bijection.**

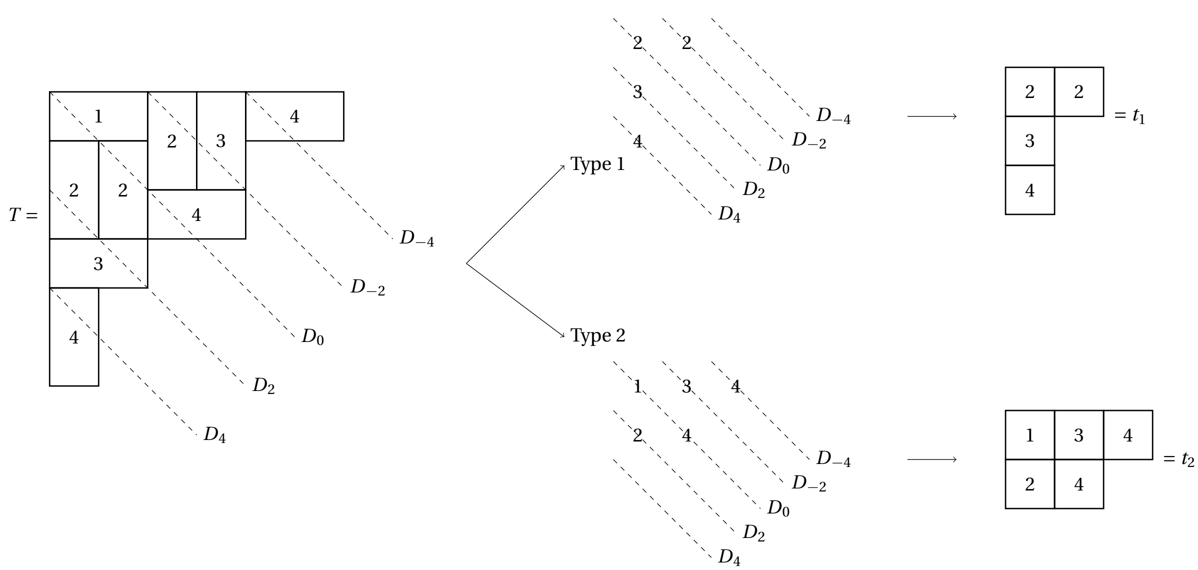
Theorem 4.12. *Let λ be a pavable partition with 2-quotient (μ,ν). The set of domino tableaux of shape λ and the set of pairs of Young tableaux of shape (μ,ν) are in bijection.*

Definition 4.13. We say that two tableaux t_1 and t_2 are of shape (μ,ν) if t_1 and t_2 are respectivelly of shape μ and ν.

Proof. The theorem 4.12 is proved by describing explicitely the algorithm Γ that sends a domino tableau to the associated pair of Young tableaux, and the inverse algorithm Γ^{-1}. Let's note that not all pairs of Young tableaux are associated to a domino tableau. We refer the reader to [2] for more historic background about the algorithm.

Γ consists on considering the diagonal reading of entries in type 1 dominoes (resp. type 2 dominoes) only. This diagonal reading corresponds to the diagonale reading of the associated Young tableau t_1 (resp. t_2).

Let's show an example of the application of the algorithm to the domino tableau T.



We leave it to the reader to verify that the 2-quotient of $sh(T) = (6,4,4,2,1,1)$ is $((2,1,1),(3,2))$, the shape of (t_1, t_2).

The inverse algorithm, Γ^{-1}, consists on constructing recursively the domino tableau of shape λ associated to a pair of Young tableaux (t_1, t_2) of shape (μ, ν), where (μ, ν) is the 2-quotient of λ. At any step, we have a pair of Young tableaux $(t_1^{(i)}, t_2^{(i)})$, of shape $(\mu^{(i)}, \nu^{(i)})$, and the associated domino tableau $T^{(i)}$, of shape $\lambda^{(i)}$. We start the algorithm with $\mu^{(0)} = \nu^{(0)} = \lambda^{(0)} = \emptyset$. The algorithm stops when $(t_1^{(i)}, t_2^{(i)}) = (t_1, t_2)$. Then we have that the domino tableau associated to (t_1, t_2) is $T^{(i)}$.

Let's describe the i^{th} step of the algorithm.

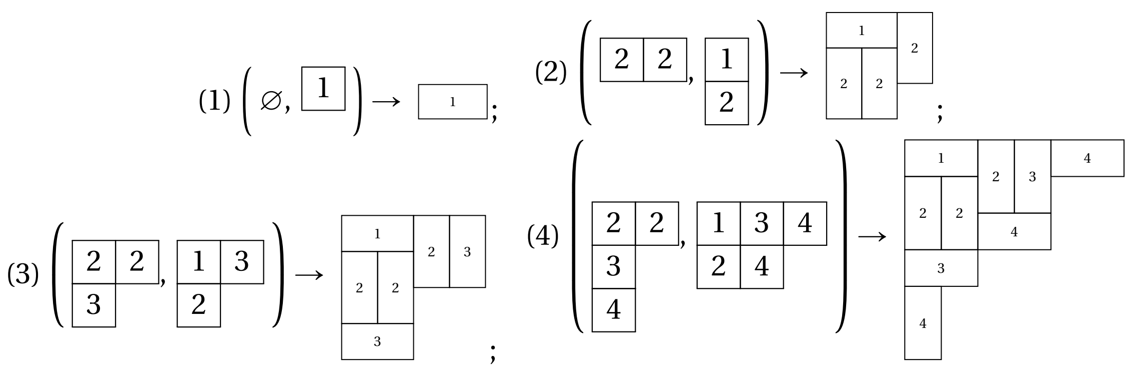
Let u_i be the smallest value appearing in (t_1, t_2) that does not appear in $(t_1^{(i-1)}, t_2^{(i-1)})$. We add to $(t_1^{(i-1)}, t_2^{(i-1)})$ all cells of (t_1, t_2) with value u_i, while preserving their original position. We get a new pair of tableaux $(t_1^{(i)}, t_2^{(i)})$ of shape $(\mu^{(i)}, \nu^{(i)})$.

To construct the domino tableau $T^{(i)}$ of shape $\lambda^{(i)}$, we follow the procedure described hereafter, starting with the leftmost diagonal.

For all cells in $t_1^{(i)}$ (resp. $t_2^{(i)}$) containing the value u_i on diagonal D_k, we add to $T^{(i-1)}$ a domino of type 1 (resp. type 2) with entry u_i on the corresponding diagonal D_{2k}. We then get the associated domino tableau $T^{(i)}$, of shape $\lambda^{(i)}$.

Here is an example of the application of the algorithm to the pair of Young tableaux

$$(t_1, t_2) = \left(\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \right).$$



The reader can notice that we found back the tableau T from the previous example.

The previous bijection is translated as follows into the symmetric function language:

Theorem 4.14. *Let λ be a partition with 2-quotient (μ, ν). Then*

$$s_\mu s_\nu = \sum_T x^T$$

where T runs over the set of domino tableaux of shape λ.

5. K-THEORY

K-theory allows us to generalize the concept of Schur functions to the stable Grothendieck polynomials. The stable Grothendieck polynomials play an equivalent role than the one of Schur functions in the K-theory cohomology ring of the Grassmanians. Roughly speaking, G_λ represents the class of the structure sheaf of a Schubert variety.

Grothendieck polynomials have first been introduced as such by Lascoux and Schützenberger, in 1982. Later on, they have been described from the combinatorial point of view by Fomin and Kirillov in 1996, and the combinatorial definition we use in this section has been introduced by Buch in 2002.

Buch also studied a bialgebra spanned by the stable Grothendieck polynomials. By taking the completion of that bialgebra, one can define a Hopf algebra. We refer the reader to [9] for further description of the links between the Grothendieck polynomials and the K-theory cohomology ring of the Grassmanians, and for further description on this Hopf algebra.

5.1. Stable Grothendieck polynomial G_λ.

This section is essentially based on [3] and discussions with the author Rebeccas Patrias. We refer the reader to [5] for more details on the history of Grothendieck polynomials.

Let's start by defining an order over sets.

Definition 5.1. Let A, B be finite sets of positive integers. Let's define an **integer set order**, denoted $\lhd$, where $A \lhd B$ if and only if $\max(A) < \min(B)$.

Definition 5.2. Let λ be a partition. A **set-valued Young tableau of shape λ** is a filling of the Young diagram of shape λ with finite, non empty sets of positive integers such that

(1) entries are weakly increasing along the rows from left to right in relation to $\lhd$.

(2) entries are increasing along the columns from top to bottom in relation to $\lhd$.

Then the polynomial x^T associated with a set-valued Young tableau T is

$$x^T = x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} x_3^{\alpha_3(T)} \ldots,$$

where $\alpha_i(T)$ is the number of occurrences of i in T.

Example 5.3. $T = \begin{array}{|c|c|c|}\hline 1,3 & 3 & 6,7 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$ is a set-valued Young tableau, and

$$x^T = x_1 x_3^2 x_4 x_6 x_7.$$

Definition 5.4. Let λ be a pavable partition. A **set-valued domino tableau of shape λ** is a paving of λ by dominoes filled with finite, non empty sets of positive integers, ordered by the integer set order such that

(1) Entries are weakly incresing along the rows from left to right.

(2) Entries are increasing along the columns from top to bottom.

Remark 5.5. Let T be a set-valued tableau. If we pick any representative of a set for each set filling T, then we have a semi-standard tableau.

We can now define the generalization of the Schur functions s_λ to the stable Grothendieck polynomial G_λ.

Definition 5.6. Let λ be a partition. The **stable Grothendieck polynomial G_λ** is defined by

$$G_\lambda = \sum_T (-1)^{|T|-|\lambda|} x^T$$

where T ranges over the semi-standard set-valued tableaux of shape λ and $|T|$ is the number of entries in T.

Remark 5.7.
- If T is a Young tableau of shape λ, then $|T| = |\lambda|$ and $(-1)^{|T|-|\lambda|} x^T = x^T$. Since the set of Young tableaux of shape λ is a subset of the set of set-valued Young tableaux of shape λ, then the terms of lowest degree in G_λ actually gives s_λ. We then have that
$$G_\lambda = s_\lambda + f(x),$$
where $f(x)$ is a symmetric function of unbounded degree where each term has degree greater than than $|\lambda|$.

Example 5.8. Let's consider $\lambda = (2,1)$. If each cell of the Young diagram of shape λ is filled only with sets containing one element, then we have Young tableaux. Otherwise, we have an infinite collection of set-valued tableaux, including the following tableaux.

$$\begin{array}{|c|c|}\hline 1 & 1,2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1,2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2,6 & 7 \\ \hline 7,8 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad .$$

Then

$$G_{(2,1)} = s_{(2,1)} - x_1^2 x_2^2 - x_1^2 x_2 x_3 + x_2 x_6 x_7^2 x_8 \pm \dots$$

An interest to studying the Grothendieck polynomials is that that proving something for G_λ automatically makes it possible to prove also for s_λ.

Remark 5.9. The multiplication of G_λ's corresponds to multiplication in the K-theory ring of the Grassmanian.

5.2. **Multiplication of stable Grothendieck polynomials.** The generalization ot the theorem 4.12 will allows us to describe the multiplication of two stable Grothendieck polynomials in the same way that we did for Schur fonctions. It gives us the following theorem.

Theorem 5.10. *Let λ be a pavable partition that has(μ,ν) as a 2-quotient. The set of set-valued domino tableaux of shape λ and the set of pairs of set-valued Young tableaux of shape (μ,ν) are in bijection.*

Proof. To prove this theorem, we can simply describe the generalization of Γ, and of Γ^{-1}.

It is pretty straightforward to generalize Γ and Γ^{-1} to the K-theoretic approach.

The generalised algorithm, which we will denote by Γ', sends a set-valued domino tableau T of shape λ to a pair of set-valued Young tableaux (t_1,t_2) of shape (μ,ν), where (μ,ν) is the 2-quotient of λ. It works exactly the same way as Γ, with labels in the cells being sets of positive integers instead of positive integers.

In other words, in order to obtain t_1 (resp. t_2), we consider on each diagonal of T only the labels of dominoes of type 1 (resp. of type 2). The diagonal reading of those labels gives us the diagonal reading of t_1 (resp. t_2).

The tableaux obtained by the algorithm are set-valued tableaux by construction. Now lets verify that these tableaux are actually set-valued Young tableaux or, in other words, that rows are weakly increasing from left to right in consideration to the integer set order $\lhd$ previously defined and that columns are increasing from top to bottom in consideration to the integer set order.

Lets suppose that for each set filling T, we select a representative at random. By definition of $\lhd$, then the resulting tableau T' is a domino tableau. We can then apply Γ to T', and we obtain a pair of Young tableaux (t'_1,t'_2). Since the type of dominoes in T and in T' are the same, then we obtain corresponding results in (t_1,t_2) and (t'_1,t'_2).

Let $a'_{i,j}, a'_{i+1,j}$ be the labels of two ajoining cells in t'_1 (resp. t'_2), then $a'_{i,j} \leqslant a'_{i+1,j}$. Since the set representatives were chosen at rendom, then we must have that, in the corresponding cells in t_1 (resp. t_2), the set labels $a_{i,j}, a_{i+1,j}$ are such that $max(a_{i,j}) \leqslant min(a_{i+1,j})$.

We then have that rows in t_1 and t_2 are weakly increasing from left to right in consideration to the integer set order $\lhd$.

Similarly, let $a'_{i,j}, a'_{i,j+1}$ be the labels of two cells disposed one on top of the other in t'_1 (resp. t'_2), then $a'_{i,j} < a'_{i,j+1}$. Since the set representatives were chosen at random, then we must have that, in the corresponding cells in t_1 (resp. t_2), the set labels $a_{i,j}, a_{i,j+1}$ are such that $max(a_{i,j}) < min(a_{i,j+1})$.

We then have that columns in t_1 and t_2 are increasing from top to bottom in consideration to the integer set order $\lhd$.

We then proved that the tableaux obtained through the generalized algorithm really are set-valued Young tableaux. Also, since the dominos types of T' and T are the same, then the shape of the tableaux t'_1, t'_2 obtained through Γ from T' by using representatives of the set labels, and the shape of the tableaux t_1, t_2 obtained through Γ' from T are the same.

Therefore the pair of shapes (μ, ν) of (t_1, t_2) is the 2-quotient of λ, since it is the same pair of shapes than for (t'_1, t'_2).

In conclusion, we have that Γ' sends a set-valued domino tableau of shape λ to a pair of set-valued Young tableaux of shape (μ, ν), where (μ, ν) is the 2-quotient of λ.

The proof for showing that Γ^{-1}' is the reverse of Γ' is very similar and consists on detailing how it works in comparison to Γ^{-1}.

The translation into the language of symmetric functions gives us the following theorem.

Theorem 5.11. *Let λ be a partition with 2-quotient (μ, ν). Then*

$$G_\mu G_\nu = \sum_T (-1)^{|T|} x^T$$

where T runs over the set of all set-valued domino tableaux of shape λ, ans $|T|$ is the number of entries in T.

Proof. Lets consider a random monomial of $G_\mu G_\nu$ constructed from the multiplication of one monomial from G_μ, $(-1)^{|t_1|-|\mu|} x^{t_1}$, with t_1 a set-valued Young tableau of shape μ, and one monomial from G_ν, $(-1)^{|t_2|-|\nu|} x^{t_2}$, with t_2 a set-valued Young tableau of shape ν.

Then

$$(-1)^{|t_1|-|\mu|} x^{t_1} (-1)^{|t_2|-|\nu|} x^{t_2} = (-1)^{|t_1|+|t_2|-|\mu|-|\nu|} x^{t_1} x^{t_2}.$$

It is clear from the previous bijection that $x^{t_1}x^{t_2} = x^T$ where T a set-valued domino tableau of shape λ that has 2-quotient (μ, ν). Moreover, it is true for all set-valued Young tableaux t_1, t_2 of respective shape μ and ν, and T the associated set-valued domino tableau.

To determine what the sign of that term is, we have to remark that by the bijection, we have $|t_1| + |t_2| = |T|$ and $|\mu| + |\nu| = |\lambda|$, which gives us the following equation.

$$(-1)^{|t_1|-|\mu|}x^{t_1}(-1)^{|t_2|-|\nu|}x^{t_2} = (-1)^{|T|-|\lambda|}x^T.$$

Since λ is pavable, then $|\lambda|$ is even and has no impact on the sign. Therefore, we have the result.

6. Word insertion through the RSK algorithm, Hecke insertion and unique rectification target

6.1. RSK algorithm and Knuth equivalence classes.

Definition 6.1. A **word** is an finite sequence of integers allowing repetitions. If any present integer appears only once in the word, then we call that word a **permutation**. A **letter** is an integer in the integer sequence that forms a word.

Definition 6.2. We say that the **length of a word w** is the number of letters in w, denoted $|w| = n$.

Definition 6.3. The **Robinson-Schensted-Knuth algorithm, or RSK,** is a well known bijection between words and pairs consisting of a semi-standard tableau and a standard tableau of the same shape that allows us to do an insertion procedure on the word to obtain a Young tableau, while keeping track of the order the cells are created in gives the standard tableau. It is possible to consult the Section 7.11 of [1] for a full description of the algorithm.

We will simply give an idea and an example of the algorithm here.

Let's say we want to insert a given word w. At every step i of the algorithm, the goal is to insert the i^{th} letter of the word w into the Young tableau obtained at the end of the last step, in order to end with another Young tableau. This Young tableau is called the insertion tableau of the step, usually denoted $P(i)$. For the first step, we insert the first letter into the empty tableau.

The i^{th} letter is inserted at the end of the first line of $P(i-1)$ only if it is greater than or equal to all the entries on that line. Otherwise, it "bumps" the smallest greater entry that is on the first line, which is then inserted into the next row. This step ends when all bumping procedures are done.

A second tableau, called the recording tableau, records where that last new cell was created in $P(i)$, and gets a corresponding cell with entry i. This tableau is called the recording tableau of the step, and is usually denoted $Q(i)$. This tableau therefore is a standard Young tableau, which means that the entries of $Q(i)$ are $\{1,2,3,\dots i\}$.

The algorithm ends when all letters have been inserted, and the insertion and recording tableaux of the word insertion are the ones obtained at the last step of the algorithm. If $|w| = n$, then $P(n)$ is the insertion tableau of w, and $Q(n)$ is the recording tableau of w.

Example 6.4. Let $w = 24132$ be the word to be inserted. The insertion, with a view of every step, gives us the following tableaux.

$$\text{Step 1}: \varnothing \leftarrow 2 \qquad P(1) = \begin{array}{|c|}\hline 2 \\ \hline\end{array} \qquad Q(1) = \begin{array}{|c|}\hline 1 \\ \hline\end{array}$$

$$\text{Step 2}: \begin{array}{|c|}\hline 2 \\ \hline\end{array} \leftarrow 4 \qquad P(2) = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline\end{array} \qquad Q(2) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

$$\text{Step 3}: \begin{array}{|c|c|}\hline 2 & 4 \\ \hline\end{array} \leftarrow 1 \qquad P(3) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad Q(3) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\text{Step 4}: \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \leftarrow 3 \qquad P(4) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \qquad Q(4) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$$

$$\text{Step 5}: \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \leftarrow 2 \qquad P(5) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad Q(5) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$$

It is relatively easy to see that two words can happen to have the same insertion tableau. For example, the reader can verify that 132 and 123 have the same insertion tableau. One may wonder what conditions are needed for two permutations to have the same insertion tableau.

Definition 6.5. We call the classes of permutations that have the same insertion tableaux **Knuth equivalence classes**.

It has been shown that a Knuth equivalence class is stable under **Knuth relations**, which are the following:

(1) $acb \simeq cab$, with $a < b < c$,

(2) $bac \simeq bca$, with $a < b < c$.

Moreover, two permutations are **Knuth equivalent** if one can be obtained from the other by a finite sequence of Knuth relations, and if and only if their insertion tableaux coincide.

The Knuth equivalence classes are important as they take part in the proof of the Little-Richardson rule, which gives us a more general description of the expansion of the multiplication of two Schur functions, as shown in the equation 1.

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda, \tag{1}$$

where $c^\lambda_{\mu\nu}$ is the Richardson-Littlewood coefficient, which is given a combinatorial description in the Littlewood-Richardson rule.

It is possible to read more on the history of the Littlewood-Richardson rule in Appendix 1 of [1].

6.2. **Hecke insertion and K-Knuth equivalence classes.** All this can be generalized to K-theory through the Hecke insertion. This algorithm is a generalization of the RSK algorithm, and allows to insert a word into a strictly inscreasing Young tableau, with a recording tableau that is a set-valued standard tableau.

The next three sections are based on [4]. See [5] for more details about Hecke insertion, and [6] for more details about K-theory, unique rectification targets and the use of the Hecke insertion in generalizing the Littlewood-Richardson rule. See also [3] for more details on the generalization of the Littlewood-Richardson rule.

Definition 6.6. The **Hecke insertion of a word w** is a generalization of the RSK insertion of w. Let T be a tableau, and x be a letter to be inserted into a row R of T.

Lets first consider the case where $x \geqslant y$, $\forall y \in R$.

(1) If $x > y, \forall y \in R$, then we can add a cell containing x to the end of R and obtain a valid strictly increasing tableau T'. Then T' is the result of the insertion of x into the row R of T.

    When inserting the i^{th} letter of w into T ends in this way, the recording tableau, $Q(i)$, simply gets an extra box in that position with the entry i.

(2) If $x = y$, for $y \in R$, then adding a cell containing x to the end of R does not give a valid strictly increasing tableau T'. Then x is "absorbed"into the last cell of R, which already contained x. Then T is the result of the insertion of x into the row R of T.

When inserting the i^{th} letter of w into T ends in this way, the recording tableau, $Q(i)$, simply gets an extra entry i in the corresponding cell. Lets note that minor adjustments may be needed to keep $Q(i)$ a set-valued Young tableau, see example.

Otherwise, let y be the smallest integer in R that is greater than x.

(1) If we can replace y with x and obtain a strictly increasing tableau, then x bumps y and we insert y into the next row.

(2) If replacing y with x does not result in an increasing tableau, then x is "absorbed into the cell of y (does not change R), and we insert y into the next row.

Example 6.7. Lets detail the steps in the Hecke insertion of 1334223.

$$\text{Step } 1: \varnothing \leftarrow 1 \quad P(1) = \begin{array}{|c|}\hline 1 \\ \hline\end{array} \quad Q(1) = \begin{array}{|c|}\hline 1 \\ \hline\end{array}$$

$$\text{Step } 2: \begin{array}{|c|}\hline 1 \\ \hline\end{array} \leftarrow 3 \quad P(2) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array} \quad Q(2) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

$$\text{Step } 3: \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array} \leftarrow 3 \quad P(3) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array} \quad Q(3) = \begin{array}{|c|c|}\hline 1 & 2,3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Here the 3 is "absorbed"into the last cell, in order to keep $P(3)$ a strictly increasing tableau.

$$\text{Step } 4: \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array} \leftarrow 4 \quad P(4) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline\end{array} \quad Q(4) = \begin{array}{|c|c|c|}\hline 1 & 2,3 & 4 \\ \hline\end{array}$$

$$\text{Step } 5: \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline\end{array} \leftarrow 2 \quad P(5) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad Q(5) = \begin{array}{|c|c|c|}\hline 1 & 2,3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$$

$$\text{Step } 6: \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \leftarrow 2 \quad P(6) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad Q(6) = \begin{array}{|c|c|c|}\hline 1 & 2,3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array}$$

In RSK, the 2 would bump the 4, but replacing the 4by a 2 would not result in a strictly increasing tableau. Therefore, here the 2 is absorbed into the last cell of the first line, and a 4 is inserted into the next line.

$$\text{Step } 7: \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \leftarrow 3 \quad P(7) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad Q(7) = \begin{array}{|c|c|c|}\hline 1 & 2,3 & 4 \\ \hline 5 & 6,7 \\ \cline{1-2}\end{array}$$

The 3 bumps the 4, but adding a cell with a 4 at the end of the next line, as you would do in RSK, does not result in a strictly increasing tableau. Therefore, here the 4 is absorbed into the last cell of the second line.

Example 6.8. Here is a example of a Hecke insertion where some adjustment is needed in order to keep the recording tableau a set-valued standard Young tableau.

$$\text{Step 1}: \varnothing \leftarrow 2 \quad P(1) = \begin{array}{|c|}\hline 2 \\ \hline \end{array} \quad Q(1) = \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

$$\text{Step 2}: \begin{array}{|c|}\hline 2 \\ \hline \end{array} \leftarrow 1 \quad P(2) = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \quad Q(2) = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

$$\text{Step 3}: \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \leftarrow 1 \quad P(3) = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \quad Q(3) = \begin{array}{|c|}\hline 1 \\ \hline 2,3 \\ \hline \end{array}$$

Here the 1 is “absorbed”into the last cell of the first row in order to keep $P(3)$ a strictly increasing tableau. This would give us the recording tableau $Q(3) = \begin{array}{|c|}\hline 1,3 \\ \hline 2 \\ \hline \end{array}$, which is not a strictly increasing set-valued Young tableau. We then have to adjust the recording tableau accordingly.

6.3. ***K*-Knuth equivalence classes.** In K-theory, we insert words instead of partitions, which allows repetitions of letters. We would like to describe the classes of words getting the same insertion tableaux under the Hecke insertion as we did with permutations with the RSK insertion algorithm.

Let's consider the ***K*-Knuth relations**, which are an extension of the Knuth relations:

(1) $acb \simeq cab$, with $a \leqslant b < c$,

(2) $bac \simeq bca$, with $a < b \leqslant c$,

(3) $x \simeq xx$,

(4) $xyx \simeq yxy$.

Definition 6.9. A class of words that is stable under the K-Knuth relations is called a ***K*-Knuth equivalence class**.

We say that two words are ***K*-Knuth equivalent** if one can be obtained from the other by a sequence of K-Knuth transformations.

Theorem 6.10. *If w and w' are two words such that their insertion tableaux are the same under Hecke insertion, then w and w' are K-Knuth equivalent.*

Remark 6.11. The converse is false! Two words in the same K-Knuth equivalence class can have different insertion tableaux under the Hecke insertion.

Definition 6.12. Since it can be useful to know if two words are K-Knuth equivalent, it would be interesting to have a description for K-Knuth equivalent classes that have a unique insertion tableau under the Hecke insertion.

We say that an increasing tableau T is a **unique rectification target** if it is the only insertion tableau under the Hecke insertion for the associated K-Knuth equivalence class.

[4] gives a few examples of ways to fill a Young diagram in order to ensure that the filled increasing Young tableau obtained through that process is a unique rectification target. We will discuss two specific ones.

Definition 6.13. A **minimal tableau** is a tableau in which each cell is filled with the smallest positive integer that will make the filling a valid increasing tableau.

Example 6.14. $T =$

1	2	3	4	5
2	3	4	5	
3	4			
4				

is a minimal tableau.

Proposition 6.15. *Every minimal tableau is a unique rectification target.*

Definition 6.16. A **superstandard tableau** is a standard tableau where the first row is filled with with $1, 2, \ldots, \lambda_1$, the second row with $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2$, etc., where $\lambda = (\lambda_1, \lambda_2, \ldots)$ is the shape of the tableau.

Example 6.17. $T =$

1	2	3	4	5
6	7	8	9	
10	11			
12				

is a superstandard tableau.

Proposition 6.18. *Every superstandard tableau is a unique rectification target.*

7. Colored permutation insertion into a domino tableau and unique rectification domino tableau

7.1. **Domino insertion.** We would like to introduce insertion of words into domino tableaux, as a generalization of the RSK algorithm. This requires certain precautions, since in a bumping procedure, bumping a horizontal domino by a vertical domino, or the reverse, would necessarely have to change the position of the other dominoes.

Thomas Lam described an algorithm to insert colored permutations into domino tableaux in [8], but we noted errors and typos both in the unpublished version ([7]) and published version ([8]) of the paper. We will present in this section the corrected algorithm.

Definition 7.1. A **colored letter** is a positive integer with possibly a bar over it. The presence, or absence, of that bar indicate the choice of a sign for that letter. Putting a bar over a letter is considered like appointing the sign "-" to the letter. Therefore colored letters are sometimes called **signed letters**.

Definition 7.2. A **colored word** is a sequence of colored letters. A colored word is a **colored permutation** if ever letter in the word appears only once.

Example 7.3. $w = 1\bar{3}\bar{2}4\bar{6}$ is a colored permutation. $w' = 1\bar{1}\bar{2}$ is NOT a colored permutation, but it is a colored word.

We will now describe the corrected algorithm that allows us to insert a colored permutation into a domino tableau. See [8] for more details and history about this algorithm. Note that this is a correction of the algorithm found in that paper.

Notation 7.4. To describe the **position of a cell c in a tableau**, we will use the following notation: (i, j), where i indicates the column c is on, starting from the left, and j indicates the line c is on, starting from the top. This corresponds to the coordinates of a graph with origen at the top left corner of the tableau and with positive axis pointing respectively to the right and down.

Example 7.5. Let's consider the following tableau.

$T =$

1	1	2	4	4	4
2	2	3	5		
4	5	6			

The position of the cell containing 3 is given by $(3, 2)$.

Definition 7.6. This algorithm allows to insert a colored letter into a domino tableau, and therefore to insert a colored permutation into a domino tableau.

Let D be a domino tableau, with no values repeated. Let i be a value that does not appear in D and that we want to insert into D.

Let's consider the subtableau A containing values that are less than i, and C, the skew subtableau of D containing uniquely values greater than i. D should normally have been splitted effectively into A and C.

We now construct a third tableau B from A by adding a domino to A with value i :

- If i is barred, we add a vertical domino in the first column of A with entry i.
- If i is not barred, we add an horizontal domino in the first line of A with entry i.

We will now add one by one the left over dominoes from C into B, one by one, following this repositionning procedure.

Let's consider the pair (B,C). They should be overlapping in at most one domino. Let λ be the shape of B, and let j be the smallest entry in C. Let dom_j be the corresponding domino in C.

We now consider the intersection of the shape of B, λ, and dom_j.

(1) If $\lambda \cap dom_j = \varnothing$, then we set $B' = B \cup dom_j$ and $C' = C \backslash dom_j$.

(2) If $\lambda \cap dom_j = (k,l)$, a square in position (k,l), then we add a domino with value j to B such that the shape of the tableau B' obtained is $\lambda \cup dom_j \cup (k+1, l+1)$ and we set $C' = C \backslash dom_j$.

(3) If $\lambda \cap dom_j = dom_j$**, and** dom_j **is horizontal**, then we bump dom_j to the next row, and we set B' to be the union of B with an additional horizontal domino with value j in the row bellow dom_j. We set $C' = C \backslash dom_j$.

(4) If $\lambda \cap dom_j = dom_j$**, and** dom_j **is vertical**, then we bump dom_j to the next column, and we set B' to be the union of B with an additional vertical domino with value j in the column to the right of dom_j. We set $C' = C \backslash dom_j$.

This repositionning procedure is repeated with the couple (B',C'), until the skew tableau C becomes empty.

The resulting B tableau is the insertion tableau of i into D, which we denote $D \leftarrow i$ or $D \leftarrow \bar{i}$, depending if i was unbarred or barred.

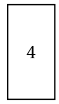
If $w = w_1 w_2 w_3 \ldots$ is a colored permutation with w_i its colored letters, then we denote $P(i)$ the insertion domino tableau of the colored permutation $w_1 w_2 \ldots w_i$, with $P(w)$ the insertion domino tableau of w.

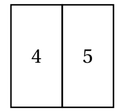
The shapes obtained at each step of the process are recorded by a standard domino tableau that we denote $Q(i)$, with $Q(w)$ the recording domino tableau of w.

Example 7.7. Hereafter, we will describe every step of the insertion of $w = 5\bar{4}\bar{6}2$.
1) $\varnothing \leftarrow 5$: We insert a horizontal domino with value 5 to the empty tableau.

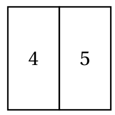
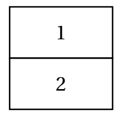
$$P(1) = \boxed{\quad 5 \quad} \qquad Q(1) = \boxed{\quad 1 \quad}$$

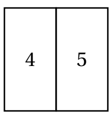
2) $\boxed{\quad 5 \quad} \leftarrow \bar{4}$: We insert a vertical domino with value 4 to $P(1)$.

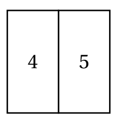
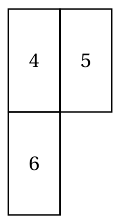
$A = \varnothing$; $B =$  ; $C =$ .

The shape of B intersects C in one cell, $(1,1)$, then by rule (2), B'=  and C'$= \varnothing$.

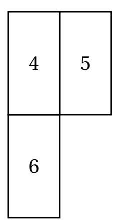
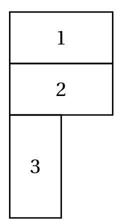
We then have that

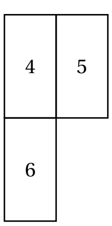
$P(2) =$  $Q(2) =$ 

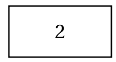
3)  $\leftarrow \bar{6}$: We insert a vertical domino with value 6 to $P(2)$.

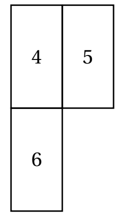
$A =$ ; $B =$  ; $C = \varnothing$.

Then

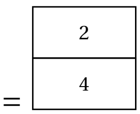
$P(3) =$  $Q(3) =$  .

4)  $\leftarrow 2$: We insert a horizontal domino with value 2 to $P(3)$.

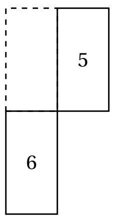
$A = \varnothing$; $B =$ 

; $C =$. 

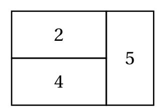
.

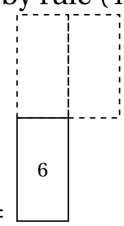
The smallest entry in C is 4, and $dom_4 \cap \lambda = (1,1)$. By rule (2), we have that B'= 

,

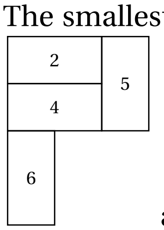
and C'= 

.

The smallest entry in C' is 5, and $dom_5 \cap \lambda$'= dom_5. Since dom_5 is vertical, by rule (4), we bump it into the column to its right and we set B”= 

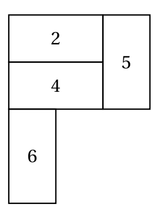
, and C”= 

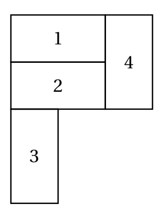
.

The smallest entry in C” is 6, and $dom_6 \cap \lambda$”= $\varnothing$. By rule (1), we then have that B”’= 

and C”’= $\varnothing$.

We then have that

$P(4) =$ 

$Q(4) =$ 

.

Remark 7.8. It is possible to describe the inverse algorithm allowing to retrace what colored permutation w was inserted from the pair of same shape domino tableaux $(P(w), Q(w))$. However we lack time and space to do so here.

7.2. **Domino "Hecke" insertion, domino insertion classes and unique rectification domino.** After studying RSK insertion and Hecke insertion, we wondered if it would be possible to have an equivalence of Hecke insertion for domino tableaux. This would allow us to insert colored words into domino tableau instead of being restricted to colored permutations. However, as it turns out, this is far from trivial. It is easy to visualise that, as in Hecke insertion, it is easy for a cell to be "absorbed" by another cell if they have the same value or in a bumping procedure. It is not so evident to see how a vertical domino can be absorbed by a horizontal domino, and vice-versa. This question needs further enquirery, and probably larger modifications than between RSK and the Hecke insertions.

We then wondered if we could put together the domino insertion algorithm and the Knuth equivalence classes and the bijection Γ that sends a domino tableau to a pair of Young tableaux, that we described previously. We first realised we could send a colored permutation onto a domino tableau using another bijection. We start with a pair of permutations, representing respectively the barred and unbarred letters of a colored permutation. We can send them onto their Young tableaux via the RSK insertion, and then we can merge those two Young tableaux into a domino tableau with the algorithm Γ^{-1}.

We wondered if the domino tableau obtained through this procedure would correspond to the one obtained through the domino insertion algorithm of the colored permutation. We realised however that this domino tableau is not at all the same than the one obtained through the insertion process, either with the shape or position of the dominoes. The reader can convince himself (or herself) by trying with the word $\bar{5}\bar{2}61347\bar{7}$ for the two techniques. Therefore, there is not a bijection between the insertion of colored permutations and the bijection described previously.

We then wondered if we could describe a domino equivalence to unique rectification targets through this bijection, and here is what we found.

Definition 7.9. We define **Knuth equivalence classes of colored permutations** by the class of colored permutations stable through Knuth transformations of the unbarred letters and Knuth transformations of the barred letters, without considering the positions of the barred letters in relation to the unbarred letters and vice-versa.

Definition 7.10. Let λ be a pavable partition. A **minimal domino tableau** of shape λ is a filling of a paving of λ such that each cell has the smallest entry possible for the domino tableau to be strictly increasing.

There are not many ways to verify that a Young tableau is a unique rectification target. There are a few techniques that are described in [4], however we were not able to prove that a minimal domino tableau splits into two unique restification targets under the Γ algorithm.

Definition 7.11. Let λ be a pavable partition, and D, a paving of this partition. We can split this paving into two Young diagrams of shape (μ,ν), where (μ,ν) is the 2-quotient of λ, by using the Γ algorithm. We can fill both these two Young diagrams in order to make them unique rectification targets, either making them minimal tableaux or superstandar tableaux for example. When we merge these two Young tableaux together by using the Γ^{-1} algorithm, the domino tableau that we get is the only domino tableau associated to the Knuth equivalence class of its filling. We call this a **unique rectification domino**.

Proposition 7.12. *For any domino shape, it is possible to find a filling of that domino shape so that the Knuth class of colored permutations associated to the filling only has one domino tableau.*

Proof. Lets remember that Γ is a bijection, that unique rectification targets are associated to a Knuth equivalence class that has a unique insertion tableau under Hecke insertion, and that Knuth equivalence classes of colored permutation are simply the pair of two Knuth equivalence classes put together. It is then pretty straightforward to see how a domino tableau constructed using the process previously is the only domino tableau associated to the Knuth equivalence class of its filling.

It is also pretty straightforward to see why it is always possible to find a filling for any given domino shape, simply by using the procedure previously described.

8. Conclusion

Studying symmetric functions, Young tableaux, domino tableaux and their K-theoretic equivalents allowed me to discover a new branch of combinatorics. I appreciated participating to the CRM summer school week about symmetric functions. I learned enourmously during thes last three weeks. I loved sharing this new knowledge to two distinct groups of students, one during the Summer student seminar of the LaCIM, and once to the Canadian Undergraduate Mathematics Conference. I then presented two different talks introducing the symmetric functions from a combinatorial point of view. This experience has been great.

It would be interesting to pursue the reserch of the questions left without answers during the stage, of which some are noted in the last section of this report.

Remerciements. I would like to thank the Laboratoire de combinatoire et d'informatique mathématique (LACIM), the Institut des sciences mathématiques (ISM), as well as my two stage directors for the summer, Rebeccas Patrias (post-doctoral researcher at UQAM) and Hugh Thomas (professor at UQAM) for their support, their enthusiasm, their empathy and great availability to meetings or answering questions.

References

[1] Stanley, Richard P. 1999. **Enumerative Combinatorics Volume 2**. Coll.Cambridge Studies in Advanced Mathematics. Éd. Cambridge University Press, New York. Sections 7.1-7.12. 585 p.

[2] Chemli, Zakaria. 2016. **Shifted domino tableaux**. *In* arXiv [En ligne] https://arxiv.org/pdf/1603.04706.pdf

[3] Patrias, Rebeccas; Pylyavskyy, Pavlo. 2014. **K-theoretic Poirier-Reutenauer Bialgebra** *In* arXiv [En ligne] https://arxiv.org/pdf/1404.4340.pdf

[4] Gaetz, Christian; Mastrianni, Michelle; Patrias, Rebeccas; Peck,Hailee; Robichaud, Colleen; Schwein,David; Tam, Ka Yu. 2015. **K-Knuth Equivalence for Increasing Tableaux** *In* arXiv [En ligne] https://arxiv.org/pdf/1409.6659.pdf

[5] Buch, Anders Skovsted; Kresch, Andrew; Shimozono, Mark; Tamvakis, Harry; Yong, Alexander 2006. **Stable Grothendieck Polynomials and K-theoretic Factor Sequence** *In* arxiv [En ligne] https://arXiv.org/pdf/math/0601514.pdf

[6] Buch, Anders Skovsted; Samuel, Matthew J. 2013. **K-theory of Minuscule Varieties** *In* arXiv [En ligne] https://arxiv.org/pdf/1306.5419.pdf

[7] Lam, Thomas. 2003. **Growth diagrams, domino insertion and sign-imbalance**. *In* Science direct [En ligne] Version publiée : http://ac.els-cdn.com/S0097316504000494/1-s2.0-S0097316504000494-main.pdf?_tid=efcd7578-7889-11e7-ac10-00000aab0f6b&acdnat=1501792129_e0921ea96f00c932dcfd07e9193ea6a8

[8] Lam, Thomas. 2003. **Growth diagrams, domino insertion and sign-imbalance**. *In* arXiv [En ligne] Version non publiée : https://arxiv.org/pdf/math/0308265.pdf

[9] Lam, Thomas; Pylyavskyy, Pavlo. 2007. **Combinatorial Hopf Algebras and K-homology of Grassmanians**. *In* arXiv [En ligne] https://arxiv.org/pdf/0705.2189.pdf

[10] **Schur polynomial**. *In* Wikipedia [En ligne] https://en.wikipedia.org/wiki/Schur_polynomial

Université du Québec à Montréal