Computability Theory

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Computable Structures

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A directed graph is computable if the edge relation is computable.

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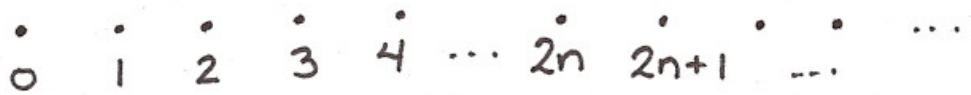

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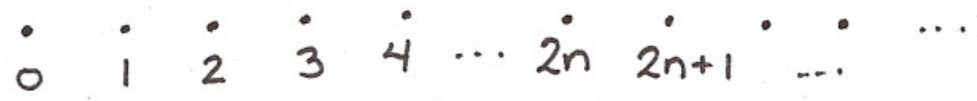

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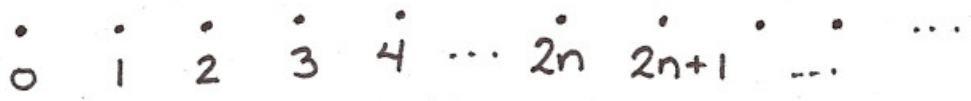

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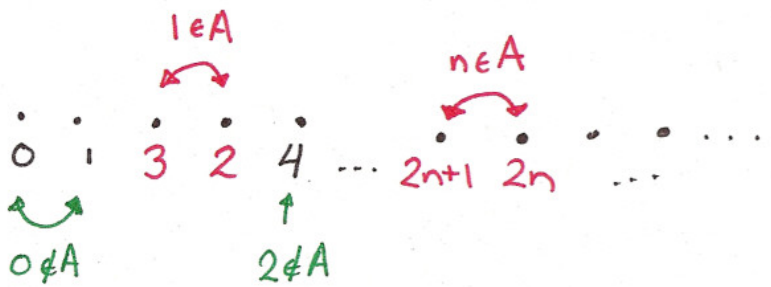

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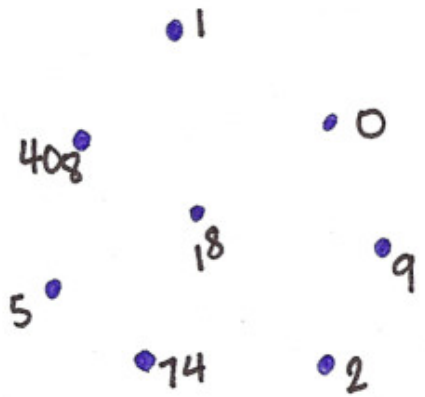
However, if the structure is trivial, such as a graph with no edges, there is no place to do coding.



Possible Degree Spectra

Theorem (Knight)

If C is an at most countable set of incomparable Turing degrees, then there is no structure whose degree spectrum is exactly those degrees computing the members of C.

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Theorem (Slaman, Wehner)

There is a structure whose degree spectrum is exactly the non-computable degrees.

Theorem (Downey-Jockusch)

If a boolean algebra $\mathcal{B}$ has a low copy, then it has a computable copy.

Degree Spectra of Boolean Algebras

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If a boolean algebra $\mathcal{B}$ has a low copy, then it has a computable copy.

Corollary

There is no boolean algebra who's degree spectrum is exactly the non-computable degrees.

Computable Categoricity

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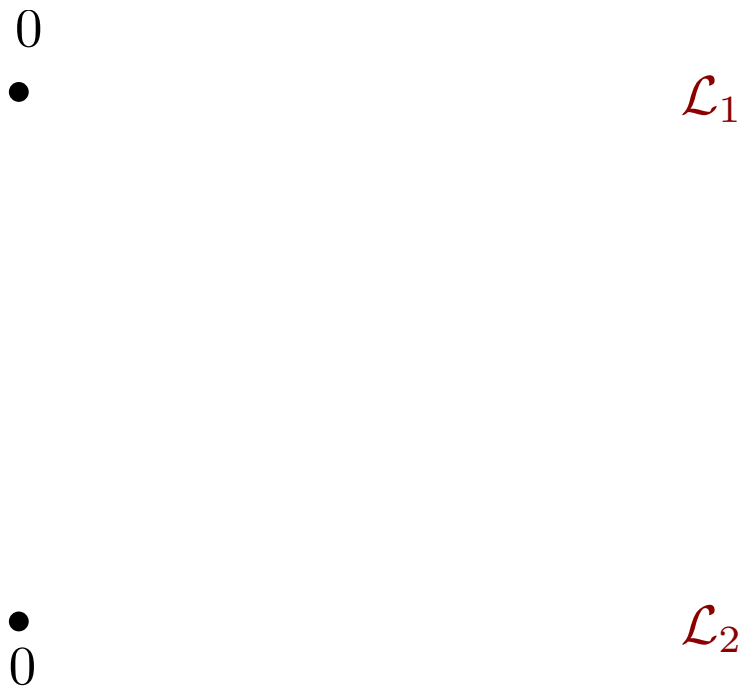

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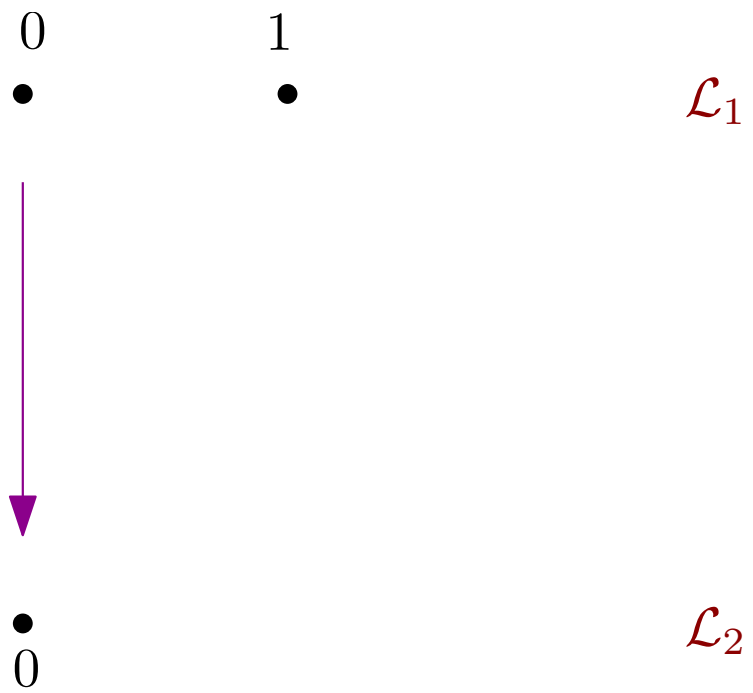

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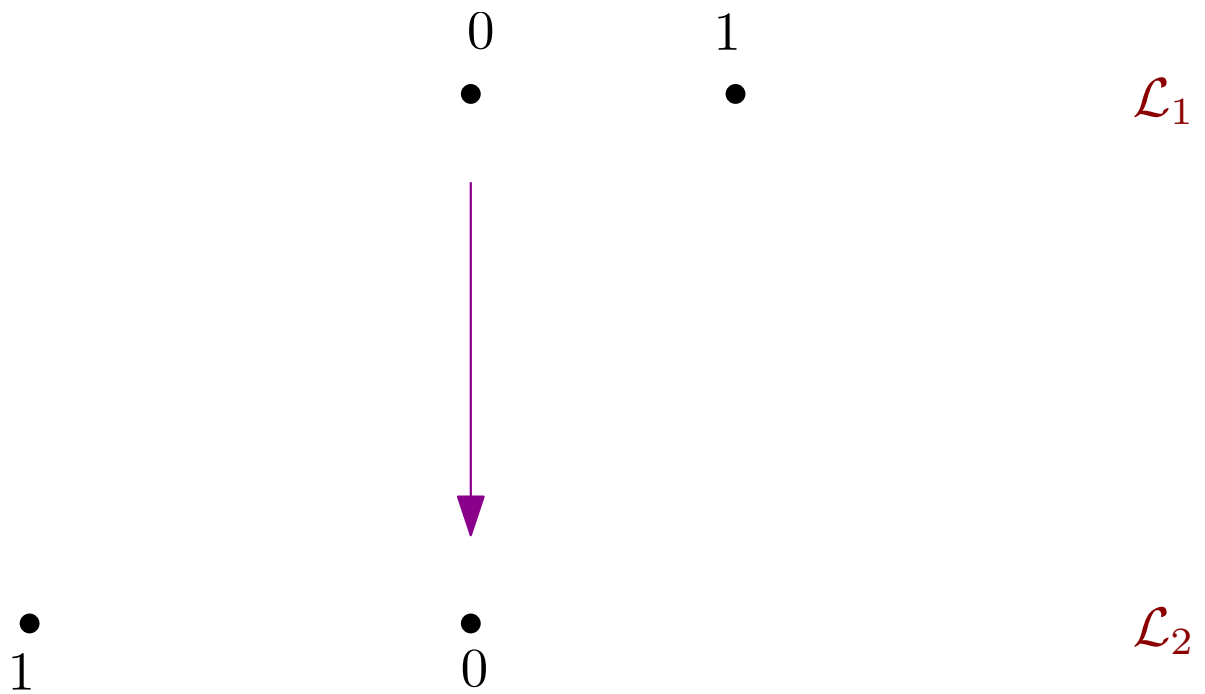

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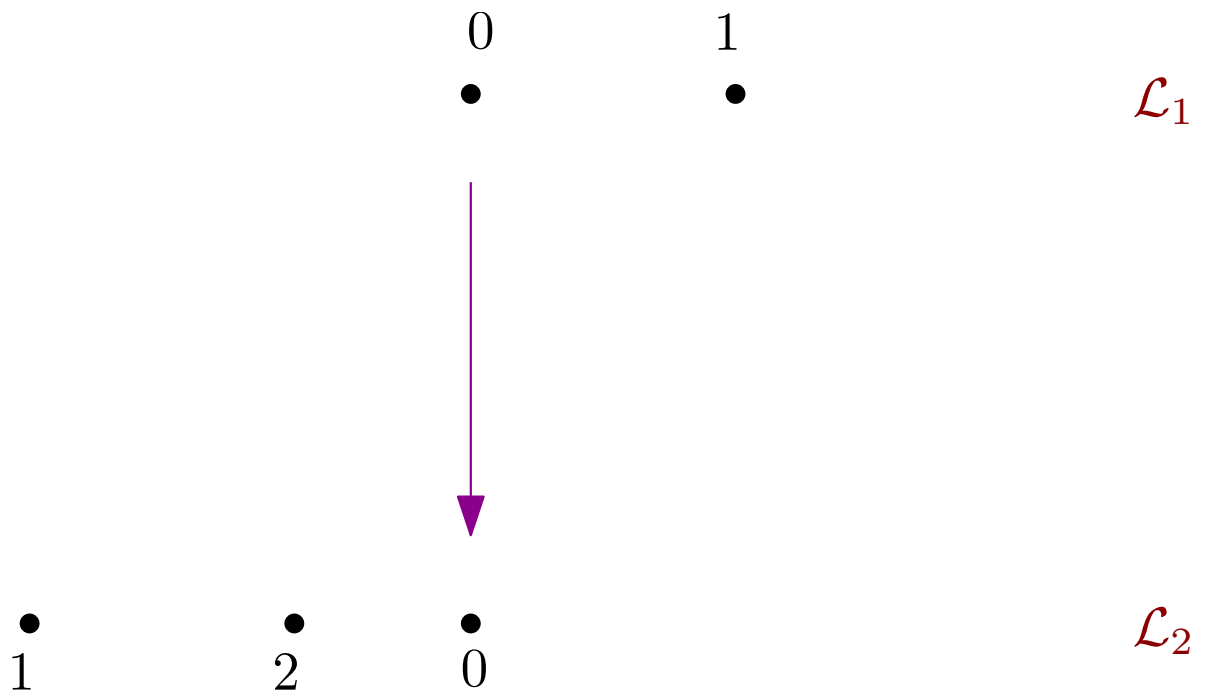

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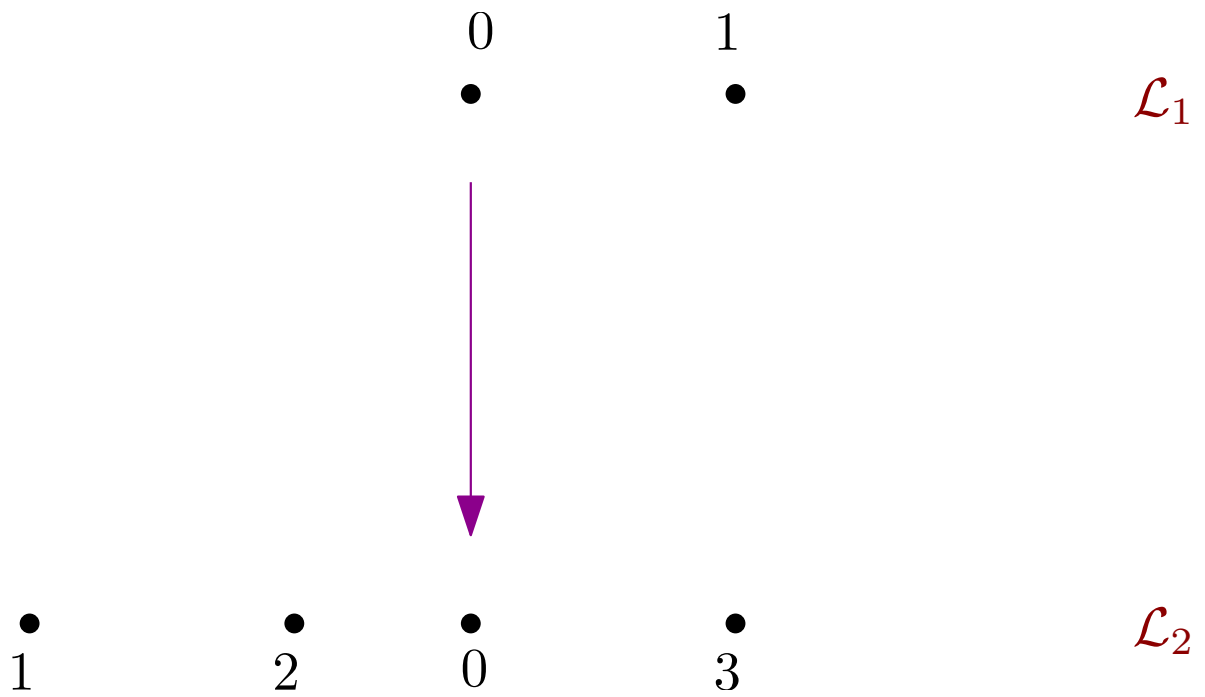

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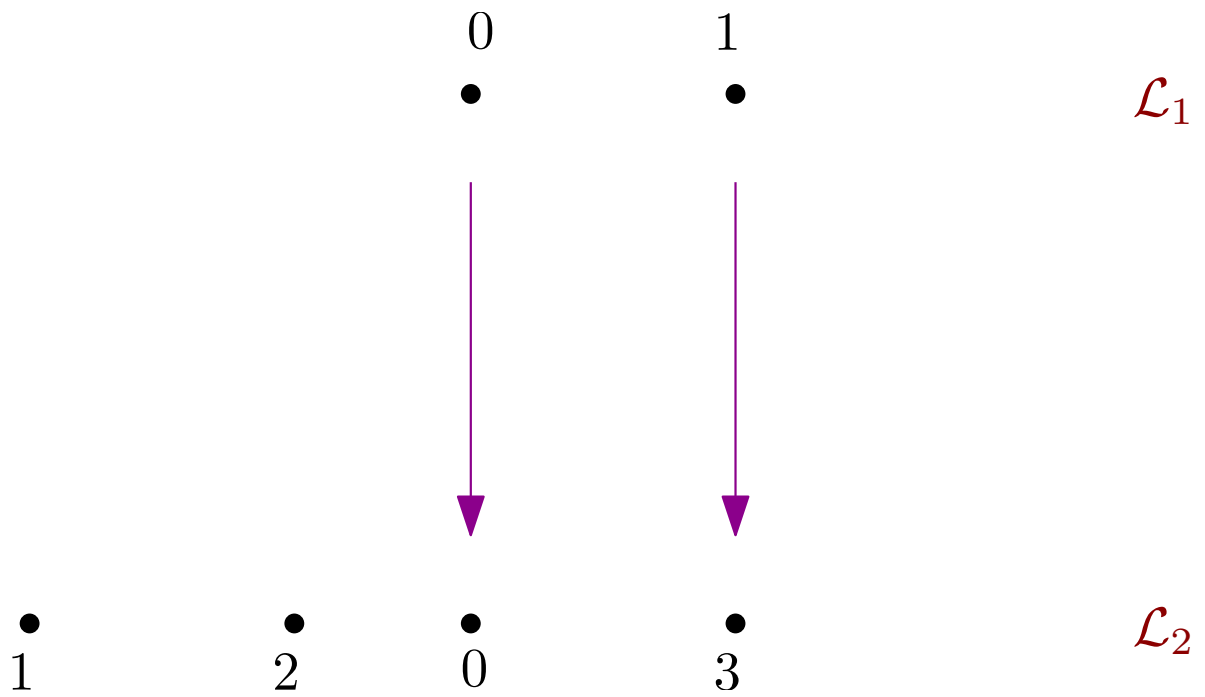

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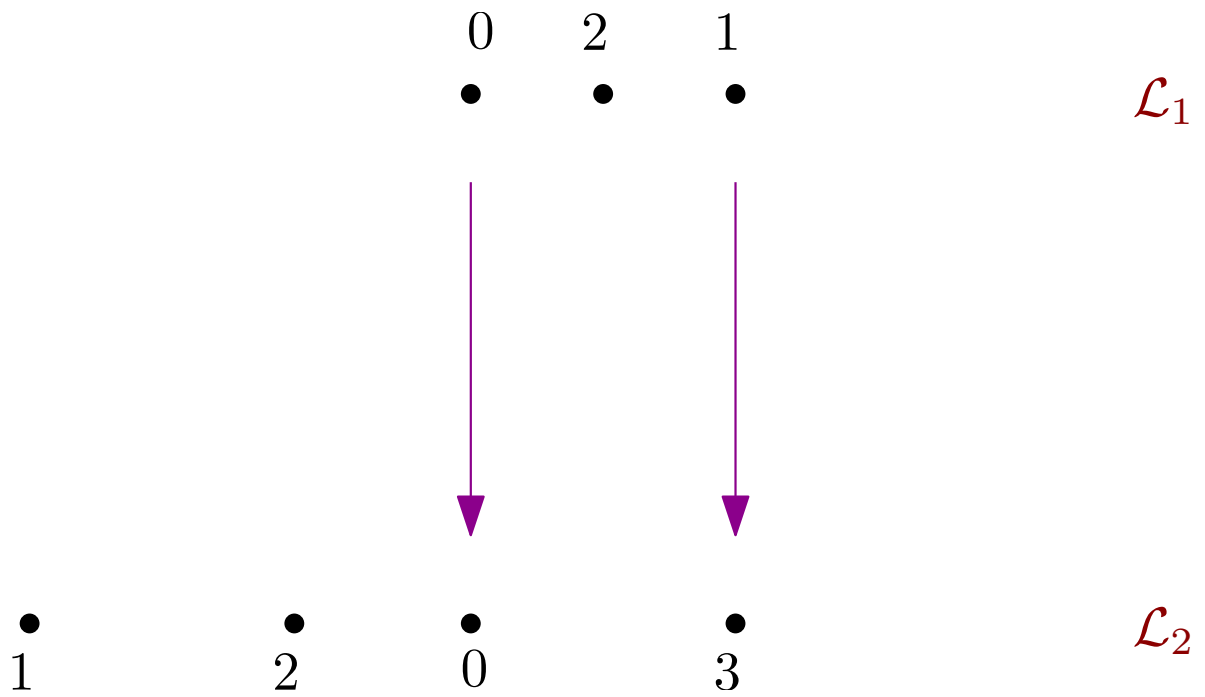

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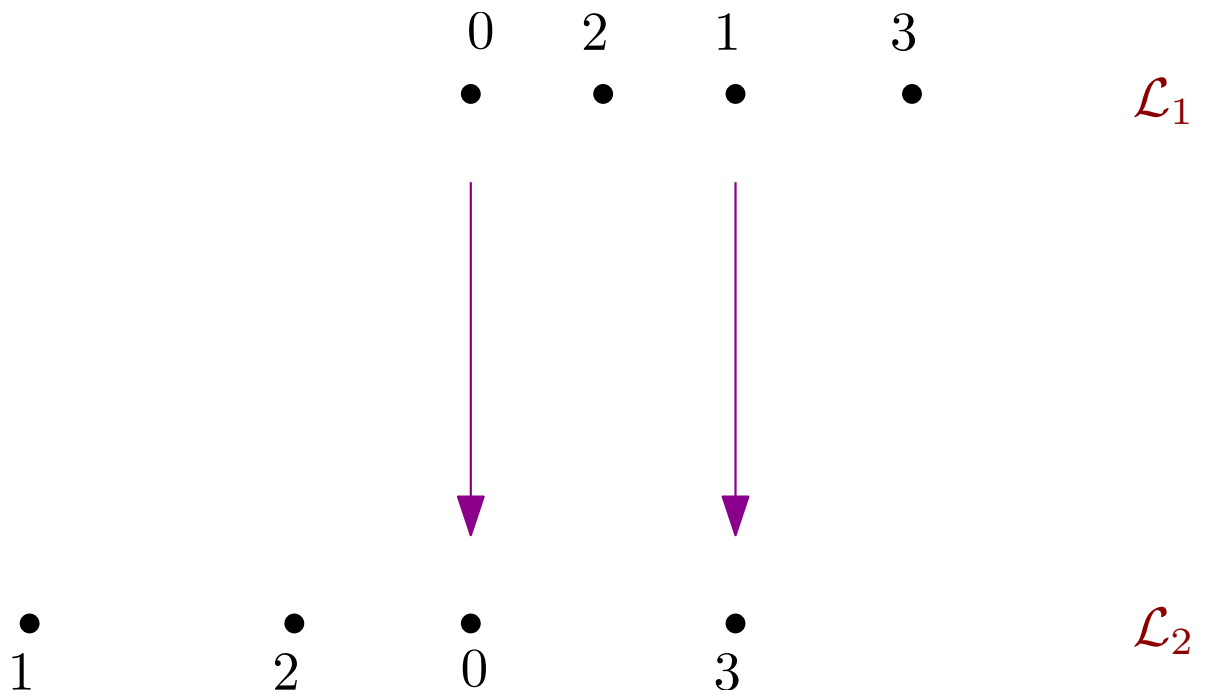

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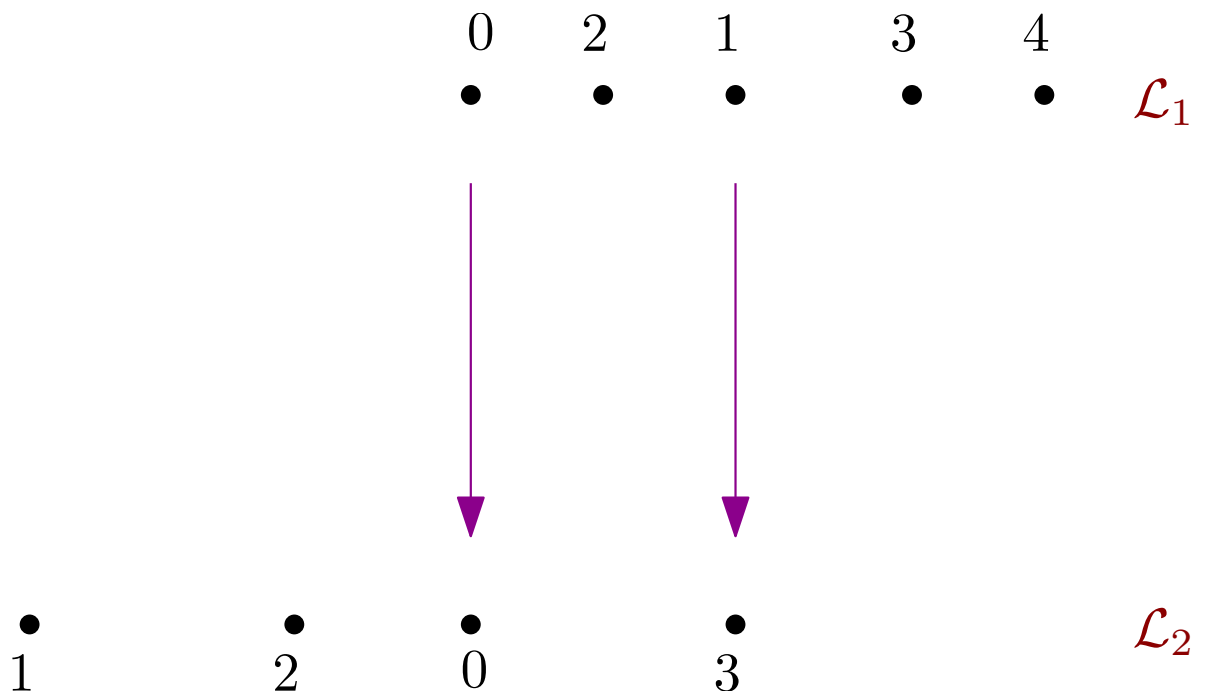

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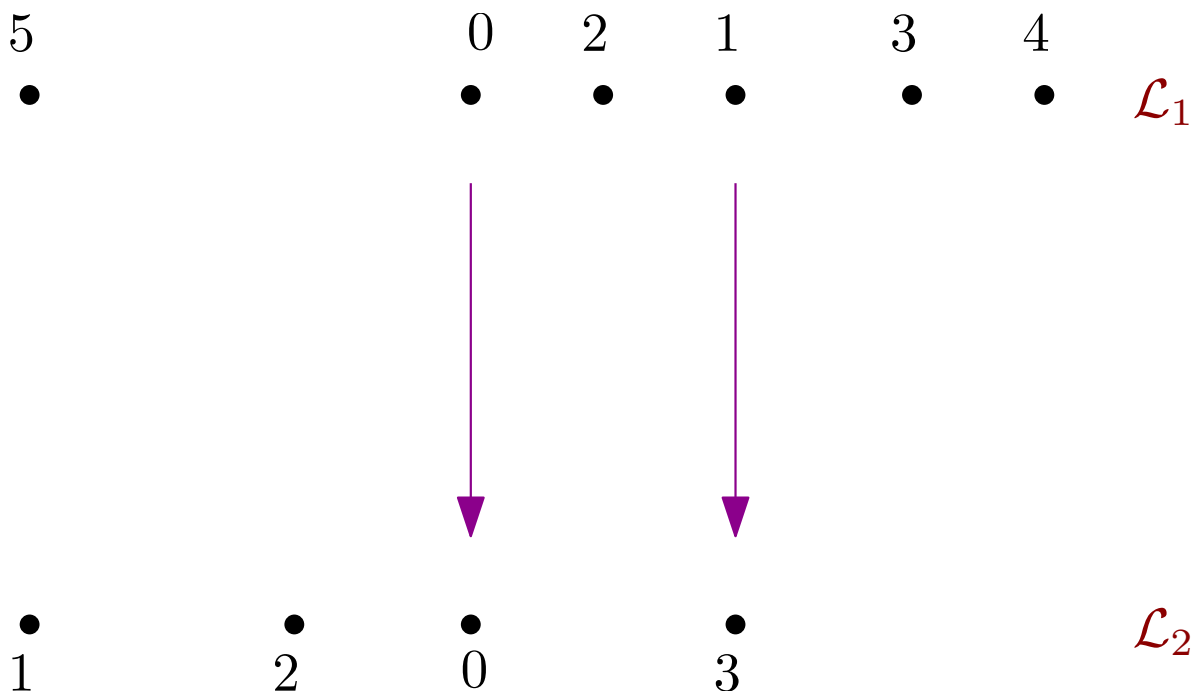

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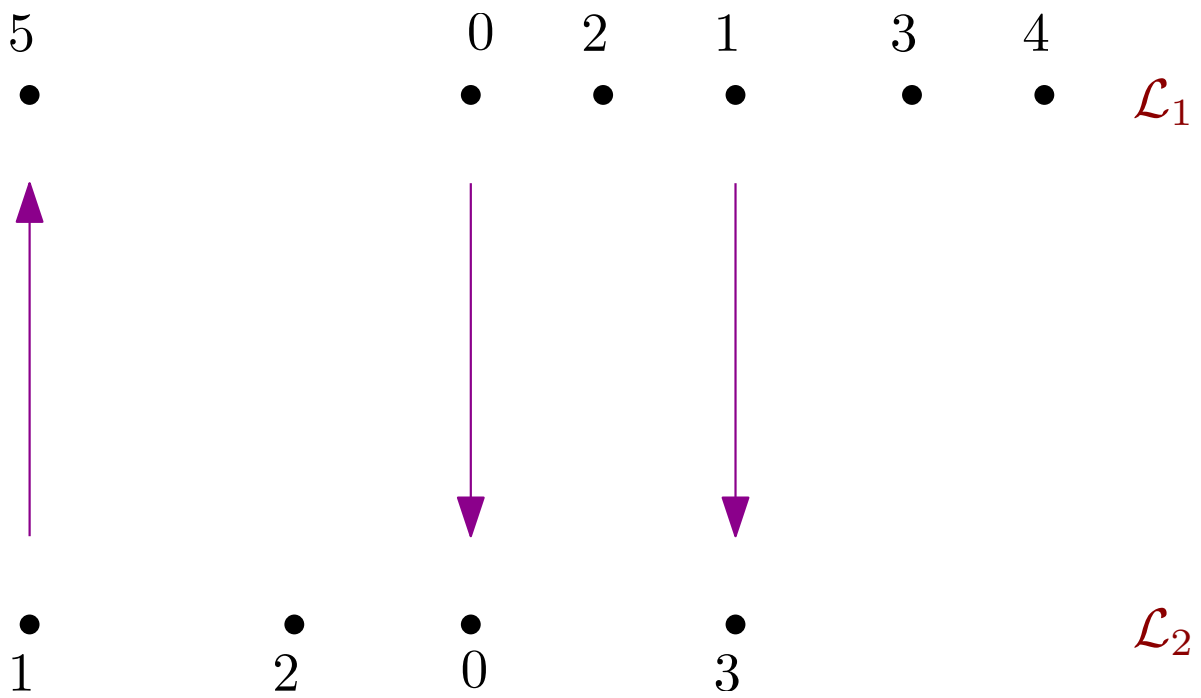

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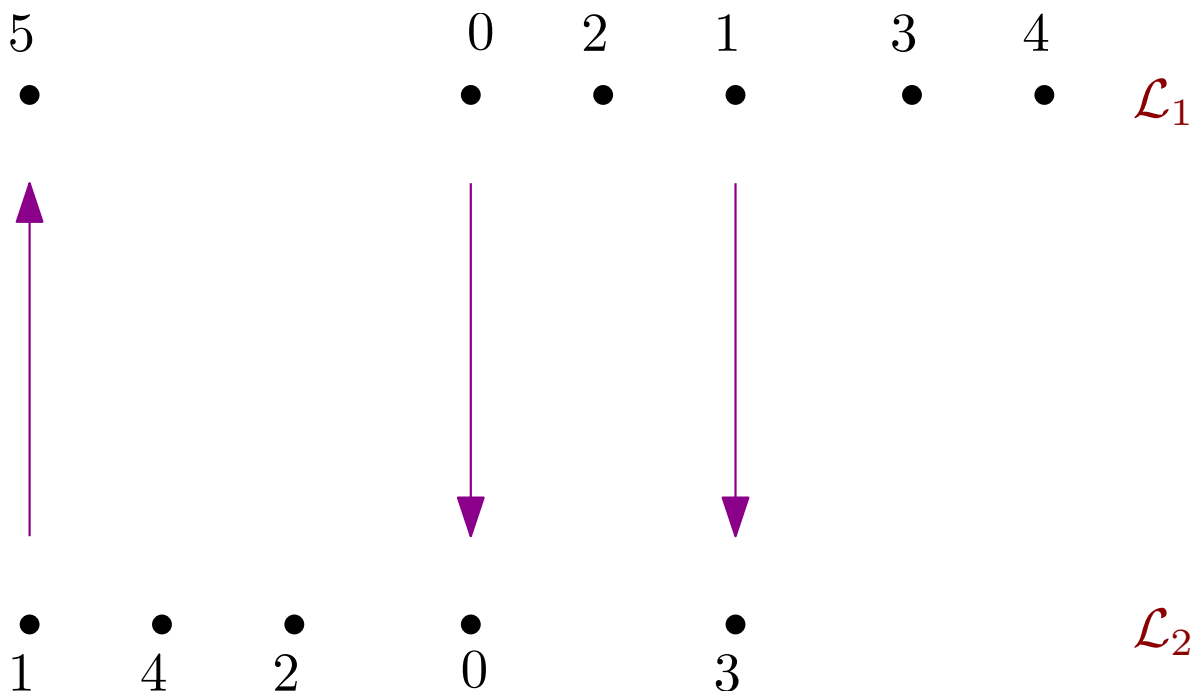

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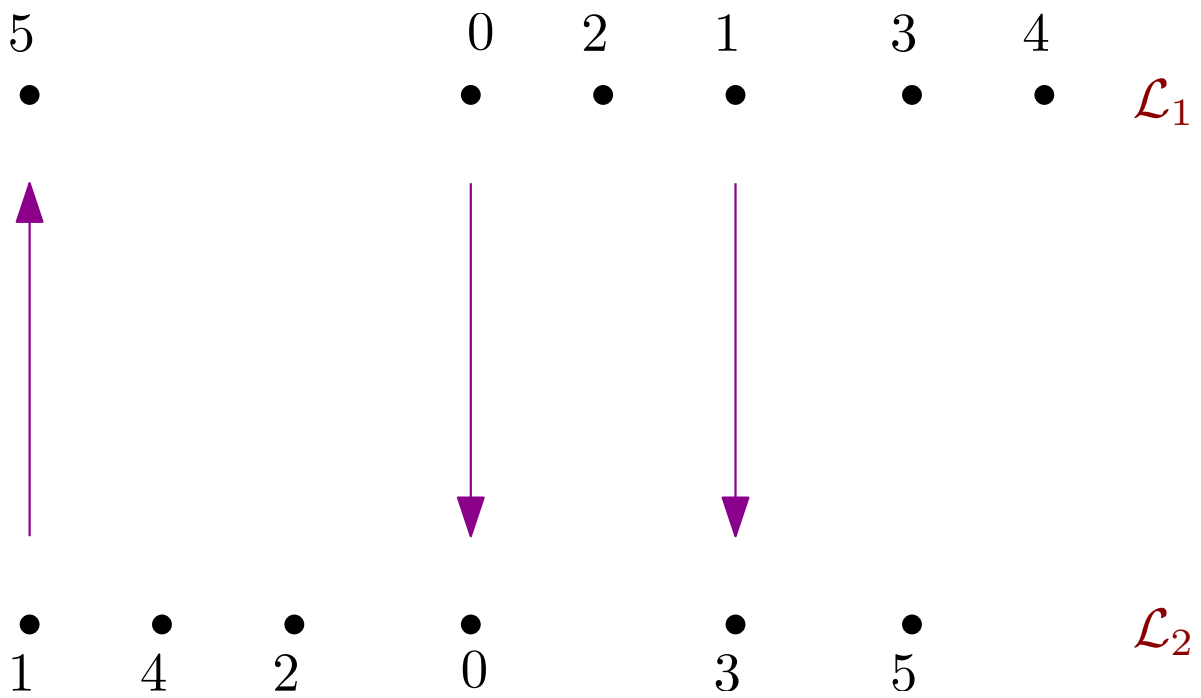

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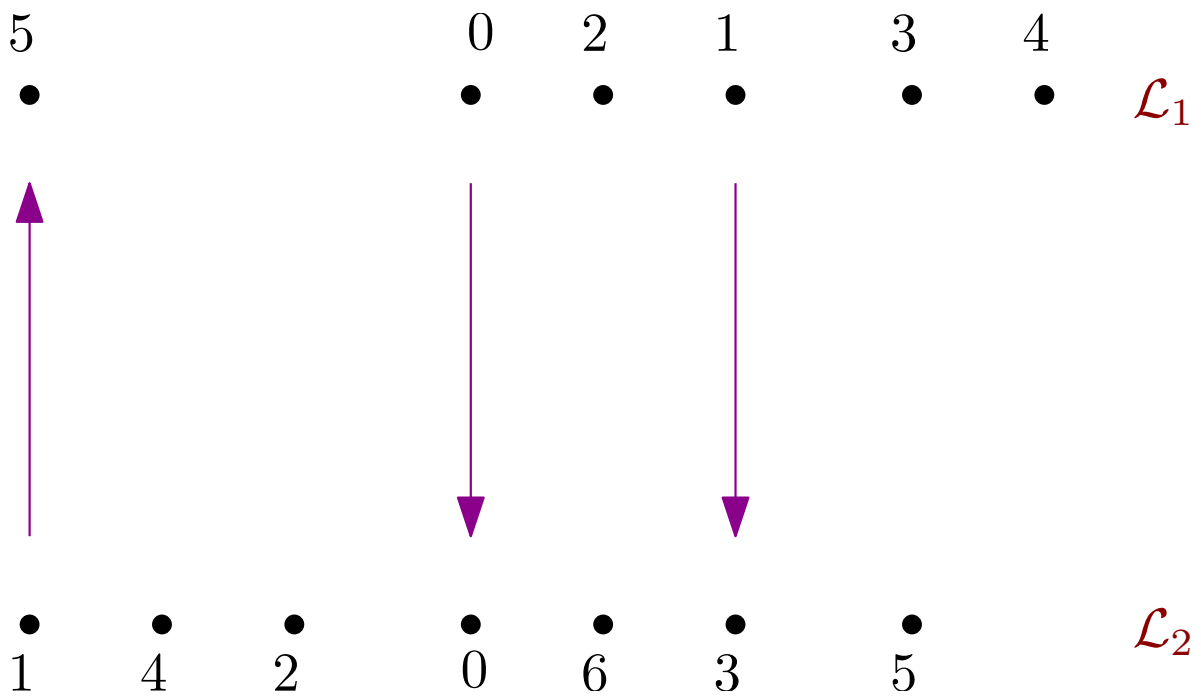

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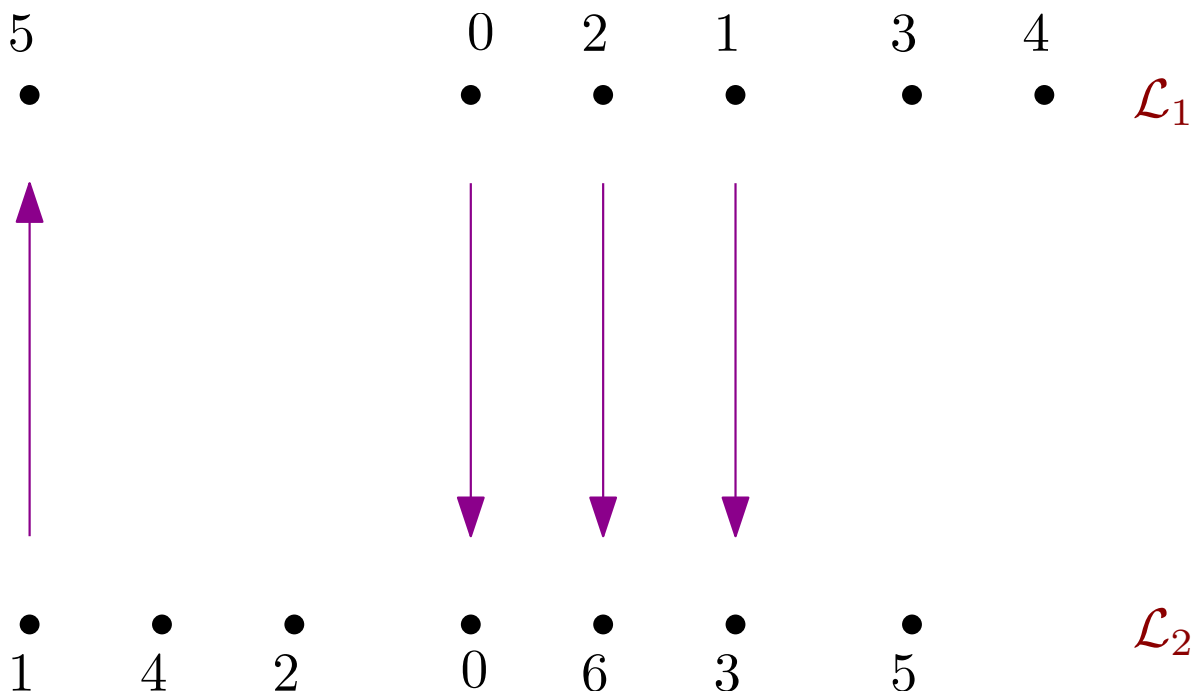

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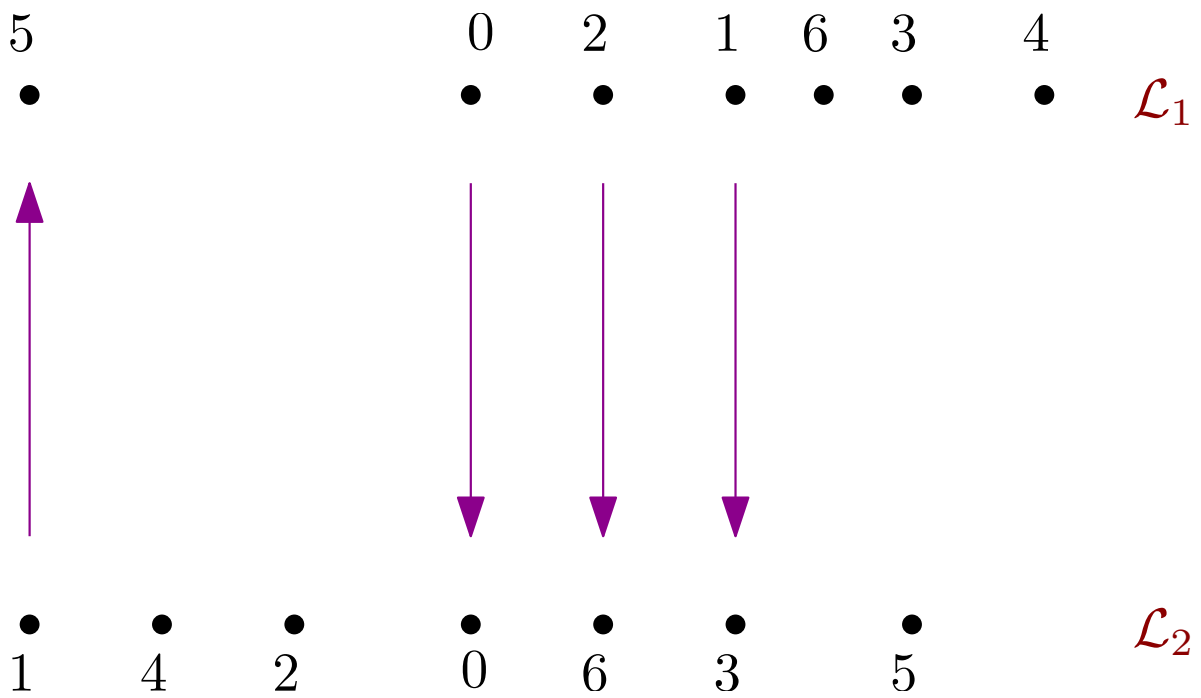

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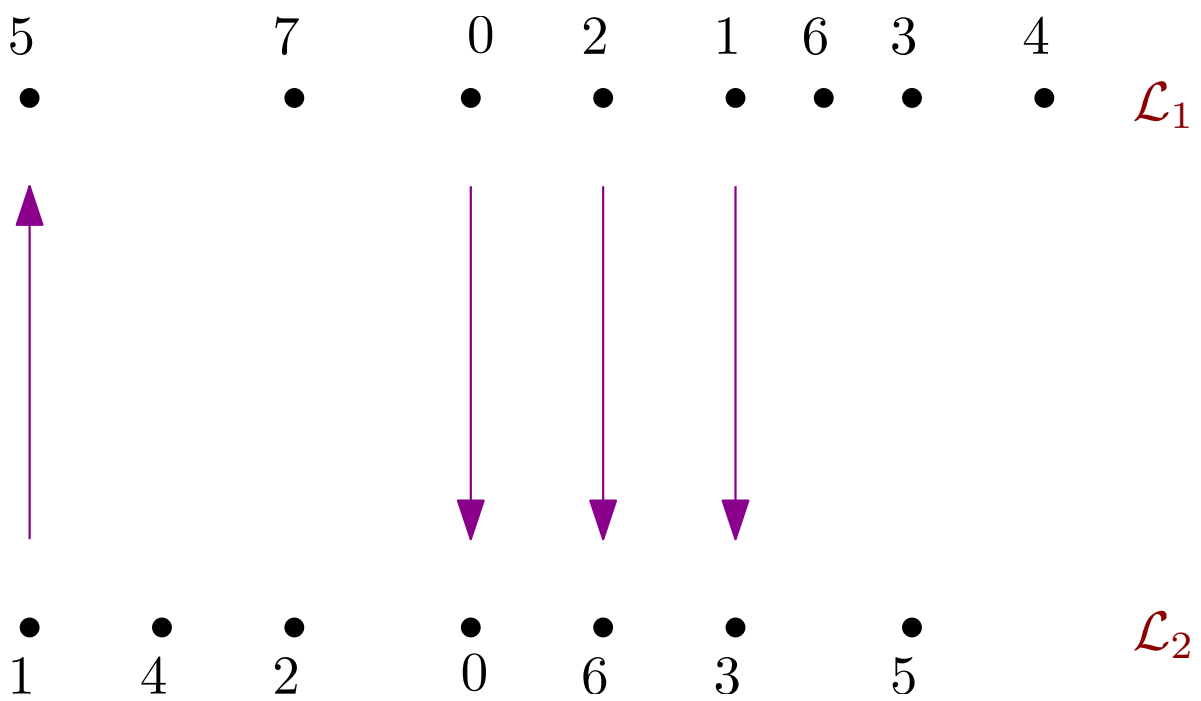

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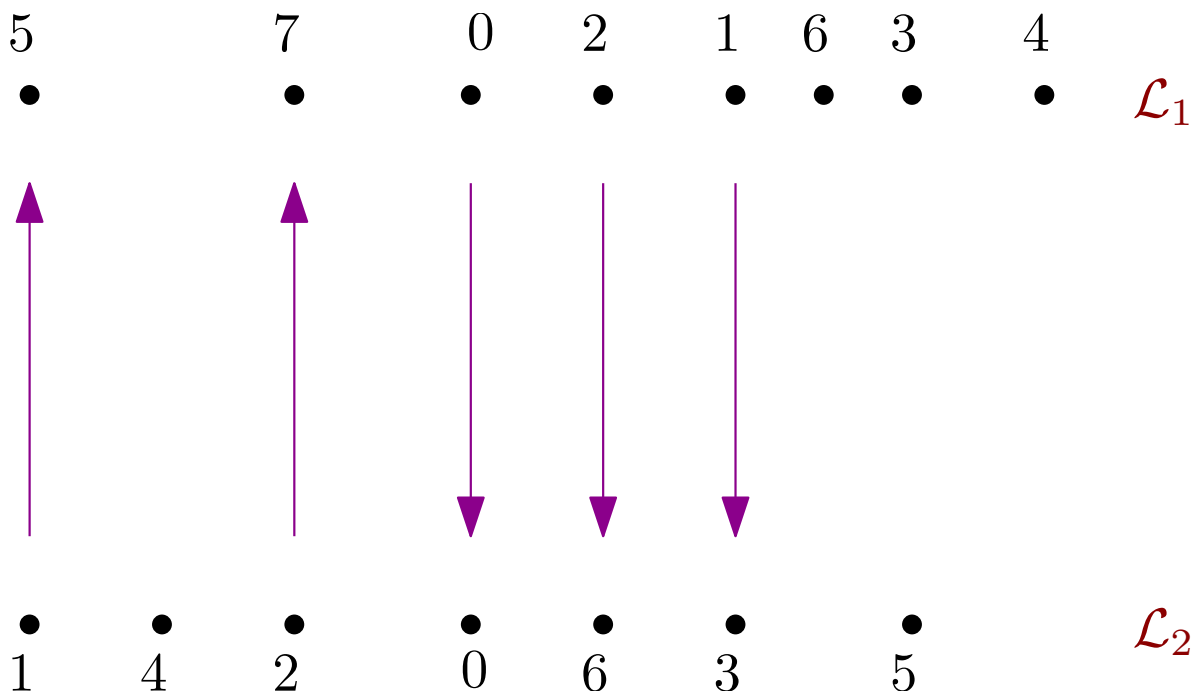

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Let $\{K_s\}_{s\in\omega}$ be a nice enumeration of $\varnothing'$ (the halting set), and consider the order $\mathcal{A}$ where the even numbers have their usual order and $2n <_{\mathcal{A}} 2s+1 <_{\mathcal{A}} 2n+2$ iff $n \in K_s - K_{s-1}$.

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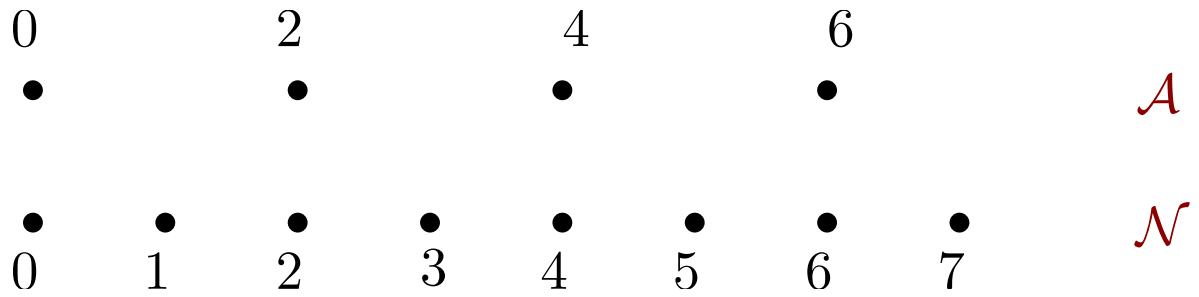

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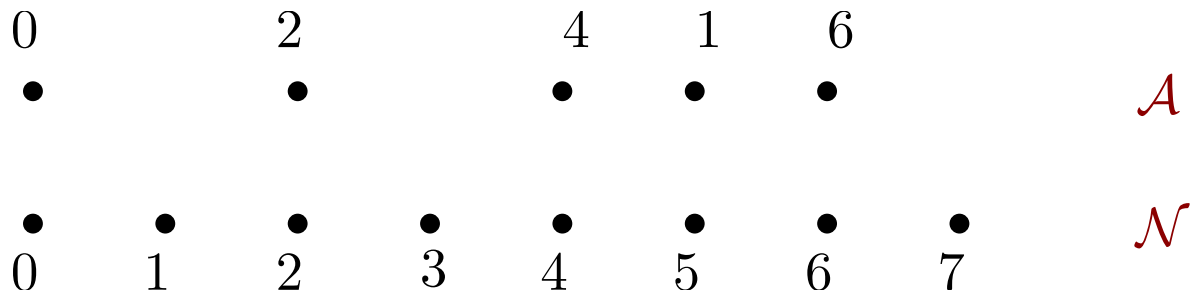

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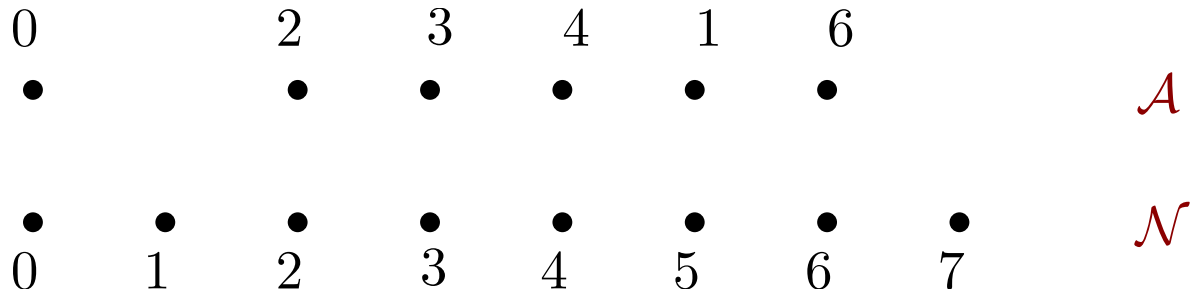

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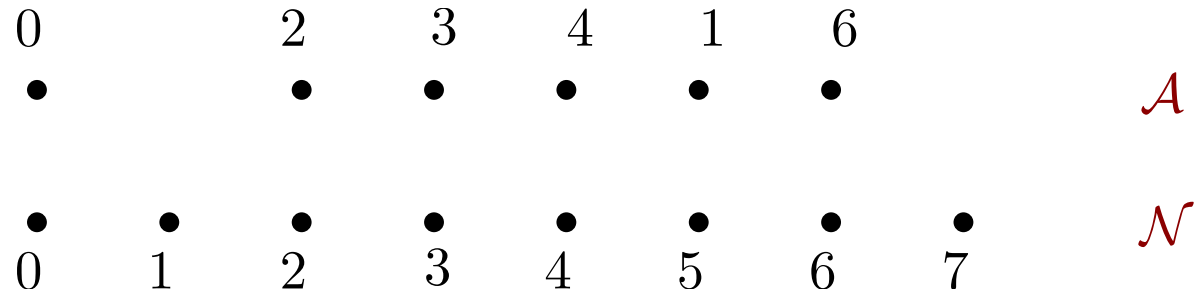

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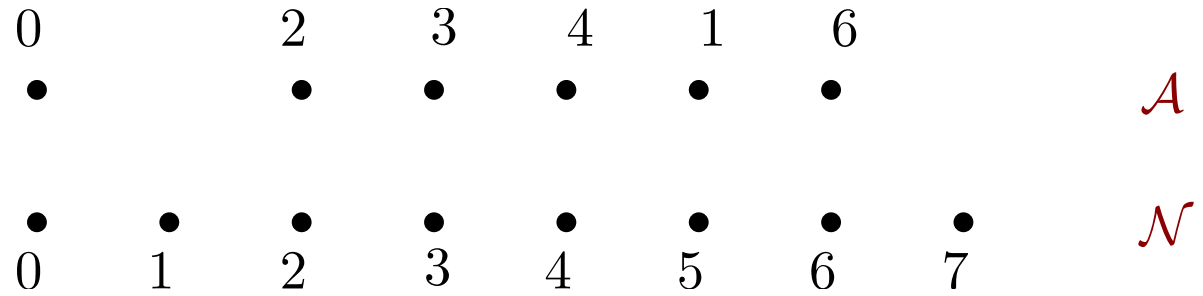

Note that any isomorphism $f : \mathcal{A} \to \mathcal{N}$ computes $\varnothing'$.
Conversely, between any two computable copies of $(\mathbb{N}, <)$, we can construct an isomorphism if we can answer existential questions about the order, hence there is a $\varnothing'$-computable isomorphism.

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Example

$(\mathbb{N}, <)$ is $0'$-**computably categorical.**

Degrees of Categoricity

Definition (Fokina, Kalimullin, Miller)

We say $\mathbf{d}$ is a degree of categoricity if there exists a computable structure $\mathcal{A}$ such that

- There exists a $\mathbf{d}$-computable isomorphism between any two computable copies of $\mathcal{A}$.
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Example

$\mathbf{0}'$ is a degree of categoricity, as witnessed by $(\mathbb{N}, <)$.

Approximations of Δ^0_2 sets

The Limit Lemma tells us that a set A satisfies $A \leq_T \varnothing'$ if and only if there exists a computable function $f(x,s)$ such that $A(x) = \lim_s f(x,s)$.

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Definition

A set A is said to be n-c.e. if there is a computable function $f(x,s)$ such that $f(x,0) = 0$, $A(x) = \lim_s f(x,s)$, and $|\{s \mid f(x,s) \neq f(x,s+1)\}| \leq n$.

Degrees of Categoricity

Theorem (Fokina, Kalimullin, Miller, 2010)

If $\mathbf{d}$ *is 2-c.e. in and above* $\mathbf{0^{(n)}}$*, then* $\mathbf{d}$ *is a degree of categoricity.*

$\mathbf{0}^{(\omega)}$ *is a degree of categoricity.*

C.e. degrees are degrees of categoricity

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In the second coding location:

(empty)

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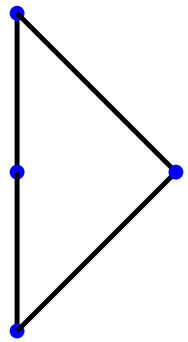

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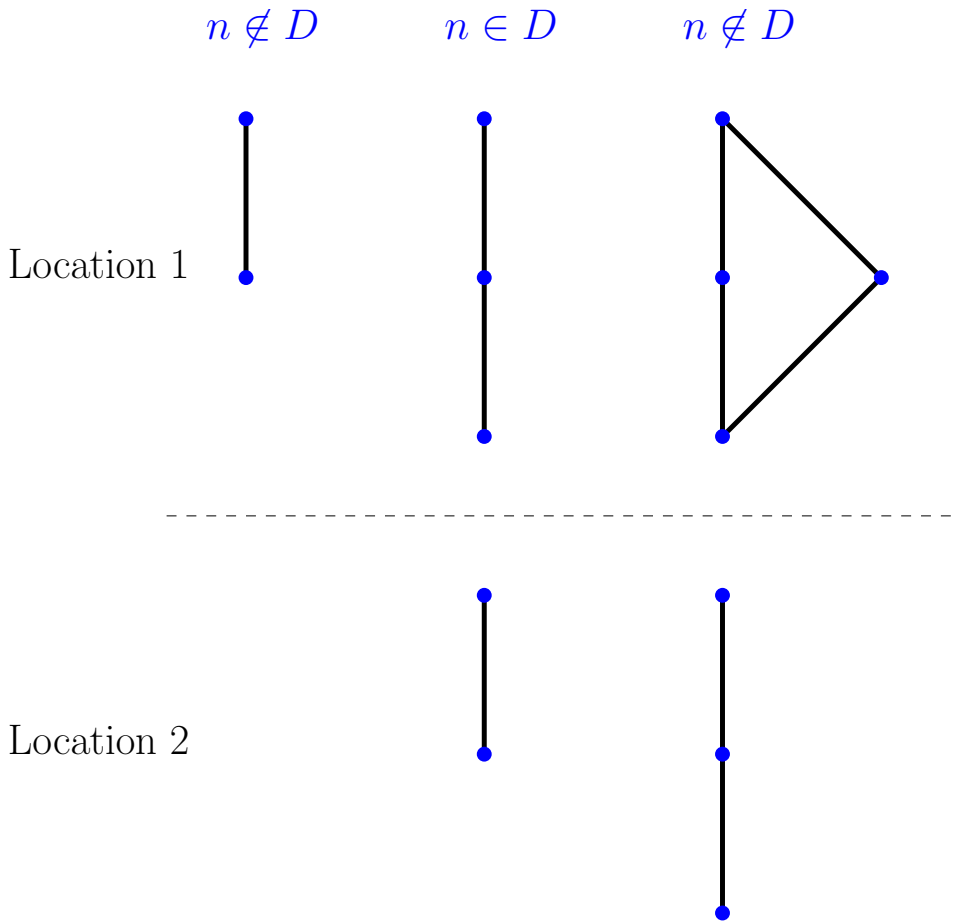

Δ^0_2 degrees are degrees of categoricity

Theorem (C, Ng)

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If we try to introduce more coding locations in the same way, we end up losing the D-categoricity of the structure.

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To code whether a particular $n \in D$, we introduce new coding locations every time there is a change in the approximation. By consulting an isomorphism on a given coding location between our distinct copies, we learn either that the approximation to D on n is correct at the stage, or that there is a further coding location introduced. This is basically like in the naive strategy.

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In order to make sure our structure is D-categorical, we keep track of all possible computable copies of our structure and build a D-computable isomorphism between them. This uses an infinite injury priority argument.

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The main trick in the basic module, is that when we "de-homogonize" a coding location, we have to be able to correctly recover a D-computable isomorphism. So we make infinite chains of nodes that look distinct stage-by-stage to control recovery, but such that they are isomorphic in the limit.

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The main trick in the basic module, is that when we "de-homogonize" a coding location, we have to be able to correctly recover a D-computable isomorphism. So we make infinite chains of nodes that look distinct stage-by-stage to control recovery, but such that they are isomorphic in the limit.

$$\cdots \to \mathbb{B}^2_\eta(\mu_L) \to \mathbb{B}_\eta(\mu_L) \to \mu_L \to \mathbb{F}_\eta(\mu_L) \to \mathbb{F}^2_\eta(\mu_L) \to \cdots$$

$$\cdots \to \mathbb{B}^2_\eta(\mu_R) \to \mathbb{B}_\eta(\mu_R) \to \mu_R \to \mathbb{F}_\eta(\mu_R) \to \mathbb{F}^2_\eta(\mu_R) \to \cdots$$