Treeability of equivalence relations via Stone duality and median graphs

Anush Tserunyan (McGill)

Joint w/ Ronnie Chen (Warsaw), Antoine Poulin (McGill), Ran Tao (Carnegie Mellon)

Standard Borel spaces.

A topological space X is called Polish if it is completely metrizable and 2^{nd} ctbl (equivalently, separable).

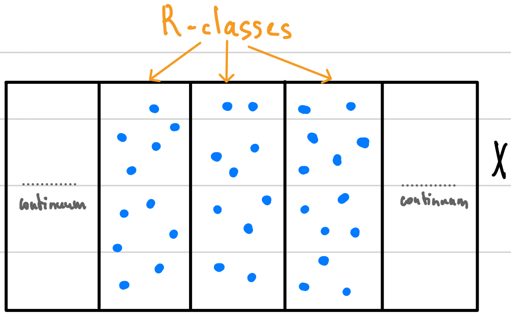
Examples of Polish spaces. $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}^{\mathbb{N}}$, $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, ℓ^p $(1 \le p < \infty)$, $L^p(\mathbb{R}^d, \lambda)$.

In these lectures, the topology on X is irrelevant, only the induced Borel sets are. All unctbl Polish spaces are Borel isomorphic, so we can pick our favourite one, like $\mathbb{R}$ or $\mathbb{N}^{\mathbb{N}}$.

Throughout, we let X be a standard Borel space, i.e. X is a set equipped with the Borel σ-algebra $\mathcal{B}(X)$ of some Polish top. on X.

CBERs and their three faces.

A ctbl Borel equivalence relation (CBER) on X is an equiv. rel. R on X such that $R \subseteq X^2$ is Borel and each R-class is ctbl.



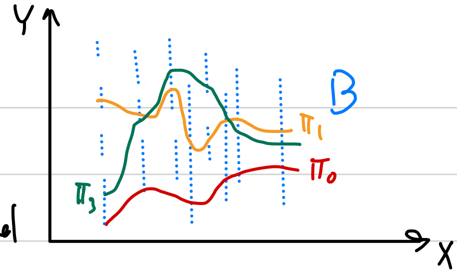
Examples.

(a) On $X := \mathbb{R}$, $R_{\mathbb{Q}}$:= rational difference, i.e. $x \, R_{\mathbb{Q}} \, y :\Leftrightarrow x - y \in \mathbb{Q}$.

(b) On $X := 2^{\mathbb{N}}$, $\mathbb{E}_0$:= eventual equality, i.e. $x \, \mathbb{E}_0 \, y :\Leftrightarrow \forall^\infty n, \; x(n) = y(n)$.

The following is the main tool from descriptive set theory for working with CBERs.

Luzin–Novikov uniformization. Let X, Y be standard Borel and $B \subseteq X \times Y$ be a Borel set with ctbl vertical fibers: $B_x := \{y \in Y : (x,y) \in B\}$ is ctbl. Then $B = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(\pi_n)$, where each $\pi_n : X \to Y$ is a Borel partial function on X.



Proof. Look up a new proof by Forte Shinko.

Use. This replaces quantifying over fibers B_x, i.e. statements like

$$\forall / \exists\ y \in Y \text{ such that } (x,y) \in B,\ (x,y) \text{ satisfies } P$$

with $\forall / \exists\ n \in \mathbb{N}$, $(x, \pi_n(x))$ satisfies P.

Face 2: group actions. Let Γ be a ctbl group and let $\Gamma \curvearrowright X$ be a Borel action. Then the orbit equivalence relation R_Γ is a CBER on X. Indeed, $x R_\Gamma y \iff \exists \gamma \in \Gamma\ \gamma \cdot x = y$.

Examples.

(a) The CBER in (a) arises from the translation action $\mathbb{Q} \curvearrowright \mathbb{R}$.

(b) The CBER in (b) arises from the translation action $\bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} \curvearrowright \bigotimes_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \cong 2^{\mathbb{N}}$.

(c) For any ctbl group Γ, $\Gamma \curvearrowright 2^\Gamma$ by shift: $\gamma \cdot x := (x(\delta\gamma))_{\delta \in \Gamma}$.

Theorem (Feldman–Moore). Every CBER R arises from a Borel action $\Gamma \curvearrowright X$ of some ctbl group Γ. In fact, Γ can be generated by involutions.

$$\begin{tikzcd} \boxed{\text{Ctbl group } \Gamma \overset{\text{Borel}}{\curvearrowright} X} \arrow[r, shift left, "\text{Orbit eq. rel. } R_\Gamma"] & \boxed{\text{CBERs}} \arrow[l, shift left, "\text{Feldman–Moore}", "\exists\, \Gamma \overset{\text{Borel}}{\curvearrowright} X \text{ with } R = R_\Gamma"'] \end{tikzcd}$$

Proof. Follows by applying Luzin–Novikov to R to obtain Borel partial functions $\pi_n : X \rightharpoonup X$, then massage these π_n into Borel bijections... $\square$

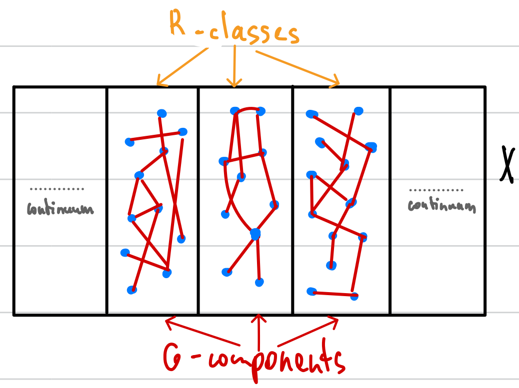
Comment. CBERs are groupoids, so the study of CBERs is generalized group theory.

Face 3: graphs. A Borel graph on X is a reflexive and symmetric Borel set $G \subseteq X^2$ (i.e. G is a set of edges on vertices X). We say that G is locally ctbl (resp. locally finite) if each $x \in X$ has only ctbly-many (resp. finitely-many) neighbours in G.

Corollary (from Luzin-Novikov). The connectedness relation R_G of a loc. ctbl Borel graph G is a CBER.

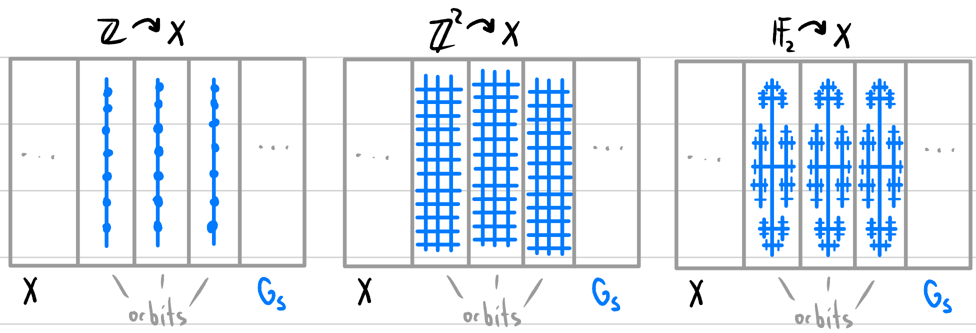
Proof. Indeed, each G-component ($= R_G$-class) is ctbl, and for all $x, y \in X$,
$$x \, R_G \, y \iff \exists n \in \mathbb{N} \; \exists (x_0, \dots, x_n) \in X^{n+1} \; (x_0 = x \text{ and } x_n = y \text{ and } \forall i < n \; (x_i, x_{i+1}) \in G).$$
The latter is Borel by Luzin-Novikov (exercise). $\square$

A graphing of a CBER R is a Borel graph G whose connected components are exactly the R-classes.



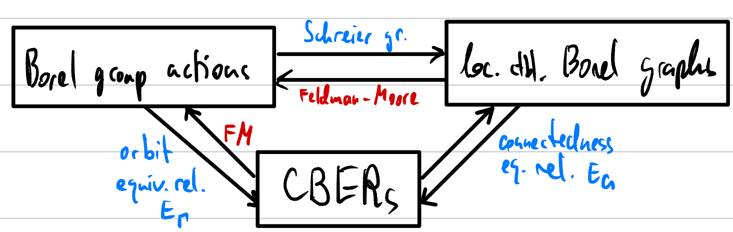
A graphing of a CBER is the analogue of the Cayley graph for groups:

Example. Let $\Gamma \curvearrowright X$ be an action of a ctbl group on a st. Borel space X. Fix a symmetric generating set S. The induced Schreier graph G_S is the graph on X where



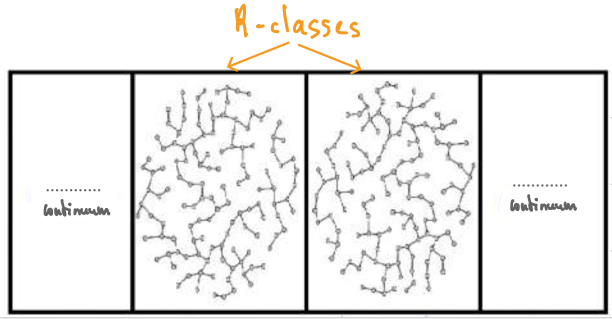
$(x, y) \in G_S \iff \gamma \cdot x = y$ for some $\gamma \in S$. By def., G_S is a loc. ctbl (finite if S is finite) graph on X.

Corollary (from Feldman-Moore). Every loc. ctbl Borel graph G on X is a Schreier graph G_S of some Borel action of a ctbl group $\Gamma = \langle S \rangle$.



Theorem (Jackson-Kechris-Louveau, Gaboriau). Every CBER admits a locally finite Borel graphing.

Treeable CBERs.



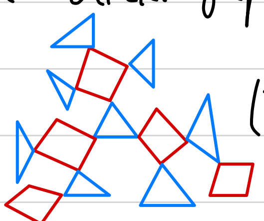
A CBER is called **treeable** if it admits an acyclic (hence minimal) graphing, called **treeing**.

Treeable CBER — by Tasmin Chu

Most CBERs aren't treeable! Those that are form a very special class analogous to free groups.

Examples. Free Borel actions of the following groups induce treeable CBERs:

(a) Free groups (The Schreier graphs are treeings.)

(b) $(\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/4\mathbb{Z})$.  (The Schreier graph can be massaged into a treeing.)

(c) Jackson–Kechris–Louveau (2002). Virtually free groups (e.g. $SL_2(\mathbb{Z})$).

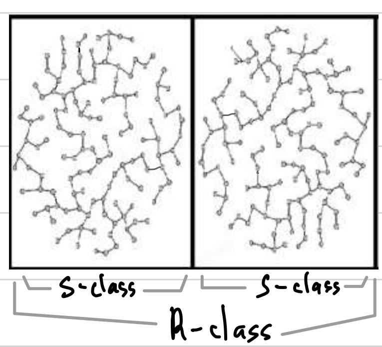
(d) Conley–Gaboriau–Marks–Tucker-Drob (2026). Surface groups, after discarding a null set.

Counter-examples.

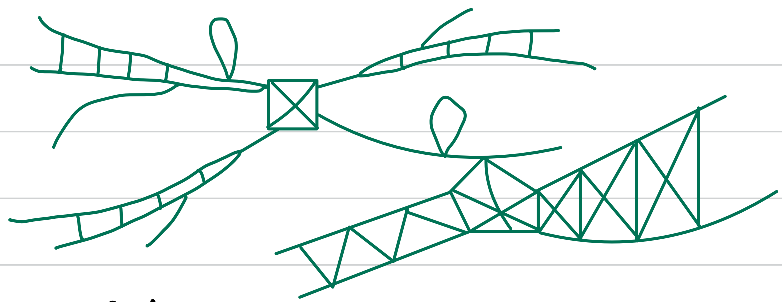
(a) Adams–Spatzier (1991). Free probability-measure-preserving actions of Kazhdan groups (e.g. $SL_n(\mathbb{Z})$ for $n \geq 3$) are not treeable.

(b) Treeable CBERs are not closed under products or ctbl increasing unions.

Finite index problem (2002). If $S \subseteq R$ are CBERs, S is treeable and $[R:S] = 2$, is R treeable?



Reformulation. R is treeable $\iff$ $\exists$ treeing T of S and Borel involution $\varphi: X \to X$ s.t. φ is an isomorphism between the two trees in each R-class.



Quasi-trees

A connected graph G is called a quasi-tree if it is quasi-isometric to a tree T, i.e.
$\exists\, f: V(G) \to V(T)$ and $C>0$ such that the f-image is C-dense in T and $-K + \frac{1}{K} \cdot d_G(u,v) \leq d_T(f(u), f(v)) \leq K \cdot d_G(u,v) + K$ for all $u, v \in V(G)$.

Example. Every f.g. Cayley graph of a virtually free group is a quasi-tree. (Exercise.)

Ghys–de la Harpe–Dunwoody. The converse also holds: if a finitely generated Cayley graph of a fin. gen. group Γ is a quasi-tree, then Γ is virtually free.

Since virtually free groups give rise to treeable CBERs (J-K-L), it's natural to ask:

Question (Tucker-Drob, 2015). If a CBER R admits a locally finite graphing whose each component is (abstractly) a quasi-tree, is R treeable?

Theorem (Chen–Poulin–Tao–Ť, 2025). Yes! More generally, if a CBER admits a locally finite graphing with tree-like components, then R is treeable.

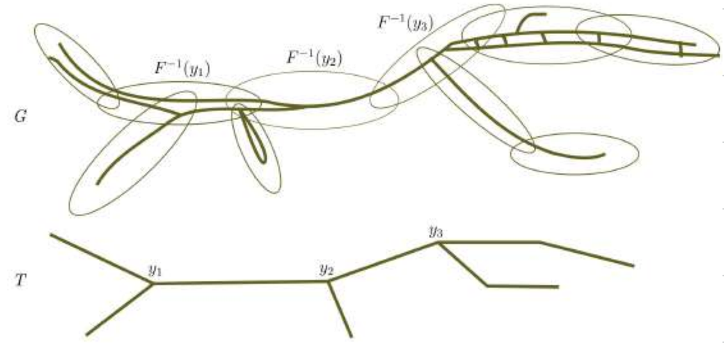
Corollary. The Jackson–Kechris–Louveau result: free Borel actions of virtually free groups are treeable (because each connected component of the Schreier graph is a quasi-tree).

Application towards the finite index problem. Instead of isomorphism φ, it is enough to build a quasi-isometry.

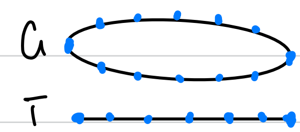
Before making "tree-like" precise, let's consider other well-known notions in this category.

Bounded tree width.

- A **tree decomposition** (Robertson-Seymour) of a graph G on X is a map $F: X \to \mathrm{Pow}(Y)$ for some tree T on Y such that
  (i) $F(x)$ is a subtree of T;
  (ii) If $(x, x') \in G$ then $F(x) \cap F(x') \neq \emptyset$.



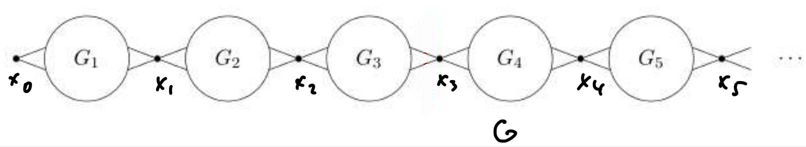
- A graph G has **bounded tree-width** if there is a tree decomp. $F: (X, G) \to (Y, T)$ with $\mathrm{TreeWidth}(G) := \sup_{y \in Y} |F^{-1}(y)| < \infty$, where $F^{-1}(y) = \{x \in X : F(x) \ni y\}$.



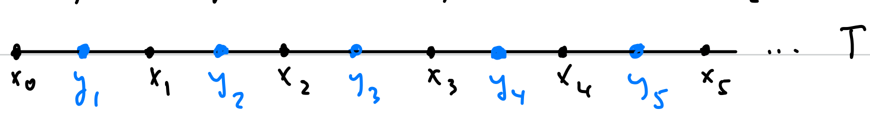
Theorem (Chen-Poulin-Tao-Ô; Jardón-Sánchez). If a CBER R admits a graphing with each connected component having bounded tree-width then R is treeable.

Applications. Used by Sole Pi to prove that all l.f. one-ended minor excluding pmp graphs are sofic. Also used by Ishikura to prove that all l.f. planar pmp graphs are sofic.

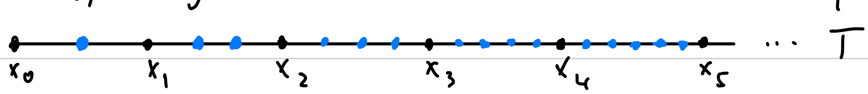
Examples.



(a) If $G_n := K_n$, a clique of size n, then G is a quasi-tree but has unbdd tree width.



(b) If $G_n := C_n$, a cycle of size n, then G is not a quasi-tree, but it has bdd tree width.



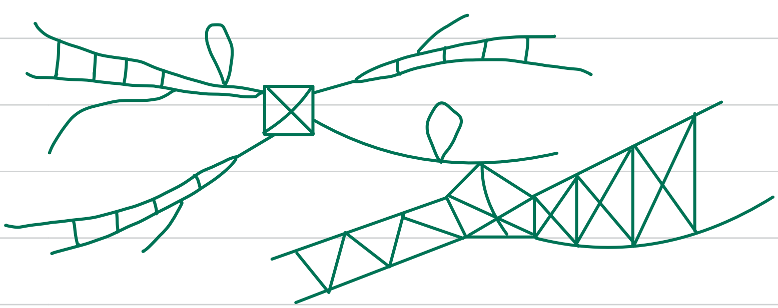
(c) If $G_{2n} := K_n$ and $G_{2n+1} := C_{2n}$, then G neither has bdd tree width nor is a quasi-tree. However, G is still tree-like and our general theorem handles this too.

No thick ends.

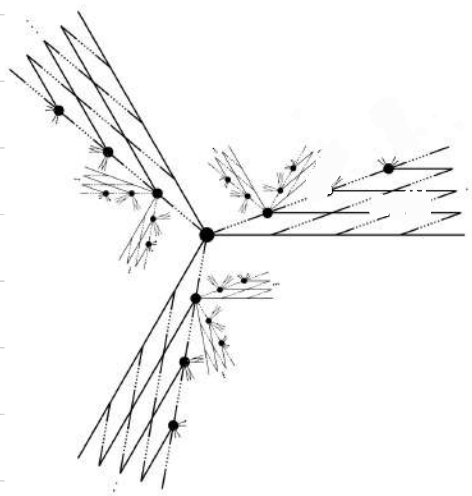
Ends of a connected graph are the "different ways to go to ∞." There are several conceptually different definitions of ends, which all coincide for locally finite graphs. Here is one of the quickest definitions (also my least favourite):

Let G be a connected loc. fin. graph on a vertex set V. An end of G is an equivalence class of rays, where we call two rays $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ end-equivalent, denoted $(x_n) \sim (y_n)$, if for each finite set $F \subseteq G$ of edges, (x_n) and (y_n) are eventually in the same component of $G \setminus F$. Denote the set of ends by $\partial G := \mathrm{Rays}(G)/\sim$. We say that a ray (x_n) converges to an end ξ if $(x_n) \in \xi$.

For an end ξ, its thickness is the sup of all $n \in \mathbb{N}$ such that $\exists$ n-many pairwise disjoint rays converging to ξ. It is Halin's theorem that thickness $= \infty \iff \exists$ ∞-many pairwise disjoint rays. An end is called thick if its thickness $= \infty$, otherwise, call it thin.



no thick ends



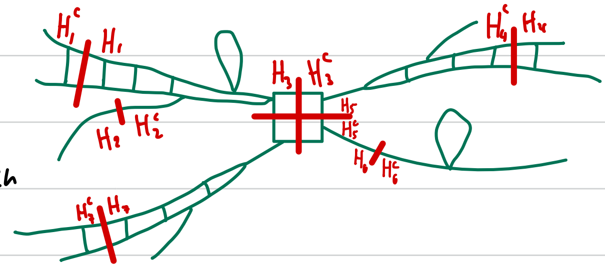
all ends thick

Corollary of our main thm (Bernshteyn + Chen-Poulin-Tao-Ô). If a CBER R admits a graphing with no thick ends, then R is treeable.

Application. Used by Hutchcroft to prove treeability of clusters in Bernoulli percolation $< p_u$.

Half-spaces/cuts and ends.

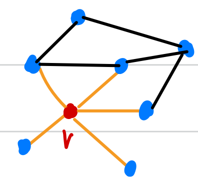
For this section, let G be a loc. fin. connected graph on a vertex set V.



— A **half-space** (or an **oriented cut**) of G is a subset $H \subseteq V$ which has finite **edge boundary** $\partial H := \{e \in G : e \text{ is between } H \text{ and } V \setminus H\}$.

Obs. (a) The set $\mathcal{H}(G)$ of all half-spaces is a (boolean) algebra, i.e. it's closed under complements and fin. unions.

(b) Every singleton $\{v\}$ is a half-space because $|\partial\{v\}| = |G_v|$ is finite by loc. finiteness.



Obs. (a) For every half-space H and end $\xi \in \partial G$,
either: every ray (x_n) converging to ξ is eventually in H, and we write $\xi \in H$
or: every ray (x_n) converging to ξ is eventually in H^c, and we write $\xi \in H^c$.

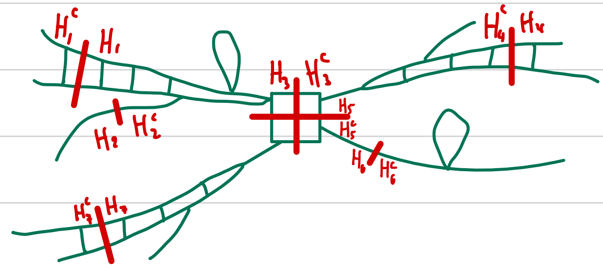
(b) For each end $\xi \in \partial G$, the set $\mathcal{U}_\xi := \{H \in \mathcal{H}(G) : \xi \in H\}$ is an **ultrafilter** on $\mathcal{H}(G)$, i.e. an upward closed collection $\mathcal{U} \subseteq \mathcal{H}(G)$ with a 0/1 law: for each $H \in \mathcal{H}(G)$, $\mathcal{U}$ contains exactly one of H, H^c. In fact, $\mathcal{U}_\xi$ is **nonprincipal** (doesn't contain finite sets).

(c) Conversely, each nonprinc. ultrafilter $\mathcal{U}$ on $\mathcal{H}(G)$ gives rise to an end $\xi \in \partial G$, i.e. $\mathcal{U} = \mathcal{U}_\xi$, so we identify ∂G = the space of nonprincipal ultrafilters on $\mathcal{H}(G)$.

Stone topology on ∂G. Each ultrafilter $\mathcal{U}$ on $\mathcal{H}(G)$ is an element of $2^{\mathcal{H}(G)}$, so $\partial G \subseteq 2^{\mathcal{H}(G)}$ and we take the subspace top, so it is generated by the sets $[H] := \{\xi \in \partial G : \xi \in H\}$, $H \in \mathcal{H}(G)$, which are relatively clopen. Observe that ∂G is a closed subset of $2^{\mathcal{H}(G)}$ because not being ultrafilter and being principal are open conditions.

Thus, ∂G is a 2^{nd} ctbl compact zero-dimensional space.

Tree-like := has a good family of cuts.

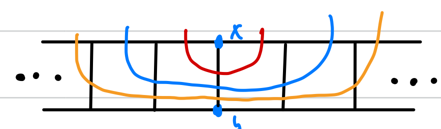


— In a (disconnected) graph G on X, call a set $H \subseteq X$ a half-space (or oriented cut) if H is a half-space of some G-component.

— In a Borel graph G on a standard Borel space X, the set $\mathcal{H}(G)$ of all of its half-spaces is a standard Borel space because it can be encoded as a Borel subset of X^2 via $H \mapsto \partial_i H :=$ the inward edge boundary of H.

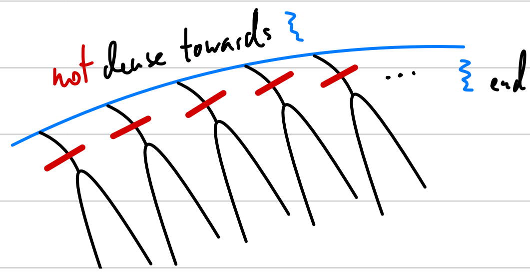
— Call a Borel graph G tree-like (Chen-Poulin-Tao-T.) if it admits a Borel family $\mathcal{H} \subseteq \mathcal{H}(G)$ of half-spaces which is

(i) finitely separating: for any two G-connected points $x, y \in X$, there are only finitely many half-spaces in $\mathcal{H}$ containing x but not y.

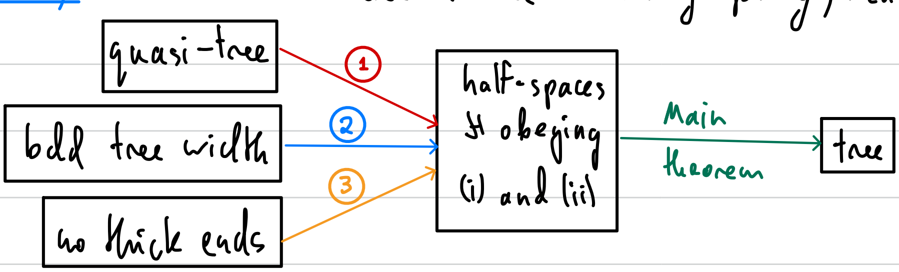


not finitely separating

(ii) dense towards ends: $\mathcal{H}$ contains a neighbourhood basis for each end of G.



Theorem (Chen-Poulin-Tao-T.). If a CBER R admits a tree-like graphing, then R is treeable.



① $\mathcal{H} :=$ all half spaces whose vertex boundary has diam $\leq C$, for some constant $C > 0$.

② $\mathcal{H} :=$ all half spaces whose vertex boundary has size $\leq C$, for some constant $C > 0$.

③ $\mathcal{H} :=$ all half spaces whose inner vertex boundary is minimal and witnesses thickness of an end.

The main theorem consists of two parts:

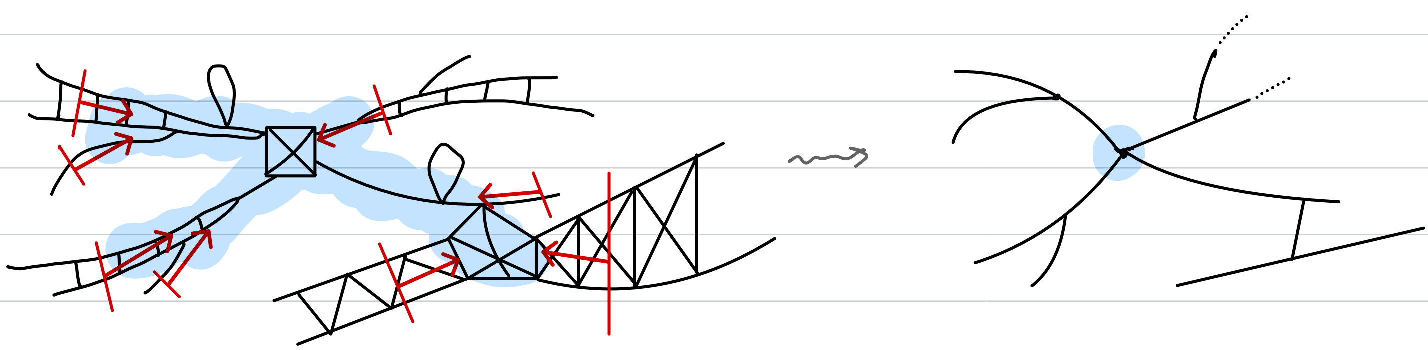
We first discuss the Stone duality part. This involves ultrafilters on a given finitely separating family of cuts...

The structure graph of $\mathcal{H}$.

Let G be a connected loc.-fin. graph on a vertex set X and let $\mathcal{H} \subseteq \mathcal{H}(G)$ be a complement-closed finitely separating (condition (i) above) family of half-spaces, including $\emptyset$ and X.

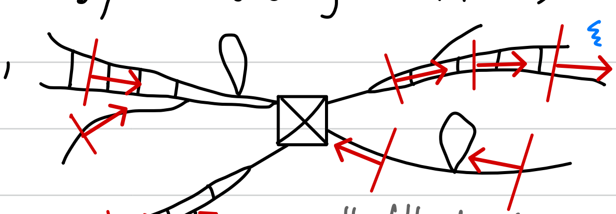
Goal. Construct a graph $M(\mathcal{H})$ which encodes the structure of $\mathcal{H}$ (and $\mathcal{H}$ carries some structural information about G). Call $M(\mathcal{H})$ the structure graph of $\mathcal{H}$.

Example.



So vertices of the structure graph should be ultrafilters on $\mathcal{H}$ (aka orientations), i.e. subsets $\mathcal{U} \subseteq \mathcal{H}$ which are upward closed ($H \in \mathcal{U}$ and $\tilde{H} \supseteq H \Rightarrow \tilde{H} \in \mathcal{U}$) and satisfy the 0/1 law (for each $H \in \mathcal{H}$, $\mathcal{U} \ni H$ or $\mathcal{U} \ni H^c$ but not both); in particular, any $H_0, H_1 \in \mathcal{U}$ intersect (if $H_0 \cap H_1 = \emptyset$, then $H_0^c \supseteq H_1$ so $H_0^c \in \mathcal{U}$ so $\mathcal{U} \ni H_0$ and $\mathcal{U} \ni H_0^c$, contradicting the 0/1 law).

Note however, that we only want those ultrafilters, which localize a region in G, as opposed to an end. In particular, we don't want ultrafilters to contain



an ultrafilter localizing an end ξ

vanishing sequences of half-spaces (i.e. decreasing sequences with $\emptyset$ intersection). More generally, we don't want ultrafilters to contain a sequence of half-spaces "converging" to $\emptyset$. To make this precise, we equip $\mathcal{H}$ with the following topology:

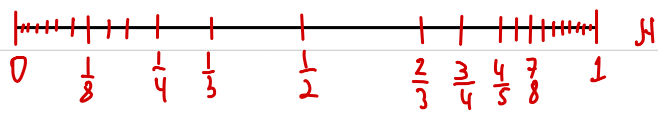
The topology on $\mathcal{H}$. Identifying $\mathcal{P}(X) \cong 2^X$, each $H \in \mathcal{H}$ is an element of 2^X, so $\mathcal{H} \subseteq 2^X$ and we equip $\mathcal{H}$ with the relative top. of the Cantor space 2^X.

Prop (finite separation via topology). Let $\mathcal{H} \subseteq \mathcal{H}(G)$ be a complement closed family with $\mathcal{H} \ni \emptyset, X$. TFAE:

(1) $\mathcal{H}$ is finitely separating.

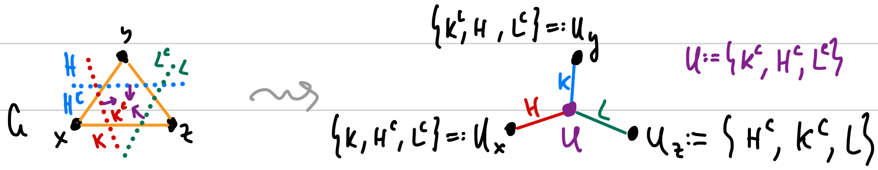
(2) $\mathcal{H} \subseteq 2^X$ is closed and every $H \in \mathcal{H} \setminus \{\emptyset, X\}$ is isolated (in other words, no $H \in 2^X \setminus \{0, X\}$ is a limit point of $\mathcal{H}$). Thus $\mathcal{H}$ looks like $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^+\} \cup \{1 - \frac{1}{n} : n \in \mathbb{N}^+\} \cup \{1\} \subseteq [0,1]$.

Proof (1)$\Rightarrow$(2). If $H_n \to H$ and $x \in H \not\ni y$, then $\forall^\infty n \; x \in H_n \not\ni y$. $\square$



We want only those ultrafilters $\mathcal{U} \subseteq \mathcal{H}$ s.t. $\emptyset \notin \overline{\mathcal{U}}$, i.e. $\exists$ open set $V \subseteq \mathcal{H}$ s.t. $\emptyset \in V \subseteq \mathcal{U}^c$. By the 0/1 law, this forces $X \in \sigma(V) \subseteq \mathcal{U}$, where $\sigma : 2^X \to 2^X$ is the bit-flip homeomorphism, so $\sigma(V) = \{H \in \mathcal{H} : H^c \in V\}$ is open. Hence, $\emptyset \notin \overline{\mathcal{U}} \iff \mathcal{U}$ is clopen in $\mathcal{H}$. Thus, we take the set $\mathcal{U}_0(\mathcal{H})$ of clopen (in $\mathcal{H}$) ultrafilters on $\mathcal{H}$ as the vertices of our structure graph $M(\mathcal{H})$.

Example.



$\mathcal{U}_x, \mathcal{U}_y, \mathcal{U}_z$ are principal, but $\mathcal{U}$ isn't.

Prop. If $\mathcal{U} \in \mathcal{U}_0(\mathcal{H})$, then every $H \in \mathcal{U}$ contains a minimal $H_0 \in \mathcal{U}$, i.e. $H_0 \in \mathcal{U}$ s.t. $\nexists H_0' \in \mathcal{U}$ with $H_0' \subsetneq H_0$. In other words, the set $\mathrm{Min}(\mathcal{U})$ of minimal elements of $\mathcal{U}$ generate $\mathcal{U}$ (by closing $\mathrm{Min}(\mathcal{U})$ upward. (If $\mathcal{H}$ consists of connected half-spaces, then $\forall \mathcal{U} \in \mathcal{U}(H)$, $\mathrm{Min}(\mathcal{U})$ generates $\mathcal{U} \Rightarrow \mathcal{U}$ is clopen.)

Lemma. If $\mathcal{U} \in \mathcal{U}_0(\mathcal{H})$ and $H \in \mathrm{Min}(\mathcal{U})$, then $\mathcal{V} := \mathcal{U} \triangle \{H, H^c\} \in \mathcal{U}_0(\mathcal{H})$; moreover, $H^c \in \mathrm{Min}(\mathcal{V})$.

Proof. Indeed, $\mathcal{V}$ is clopen because $\mathcal{U}, \{H\}, \{H^c\}$ are clopen and $\mathcal{V}$ is an ultrafilter by the minimality of H in $\mathcal{U}$: (a) if $K \supsetneq H^c$ then $K^c \subsetneq H$ so $K^c \notin \mathcal{U}$, so $K \in \mathcal{U}$ hence $K \in \mathcal{V}$.

(b) $\mathcal{V}$ still satisfies the 0/1 law. Finally, $H^c \in \mathrm{Min}(\mathcal{V})$ since if $K \in \mathcal{V}$ and $K \subseteq H^c$, then

$K^c \supseteq H$ so $K^c \in \mathcal{U}$ hence $K \notin \mathcal{U}$, so $K \in \mathcal{V} \setminus \mathcal{U} = \{H^c\}$. $\square$

So "flipping" minimal half-spaces keeps us inside $\mathcal{U}_0(\mathcal{H})$. We take such "flips" as the edges of the structure graph $M(\mathcal{H})$, i.e. put an edge $\mathcal{U} — \mathcal{V}$ if $\mathcal{U} \Delta \mathcal{V} = \{H, H^c\}$ for some $H \in \mathcal{H}$. (If $H \in \mathcal{U}$ then H is minimal in $\mathcal{U}$ and H^c is minimal in $\mathcal{V}$.)

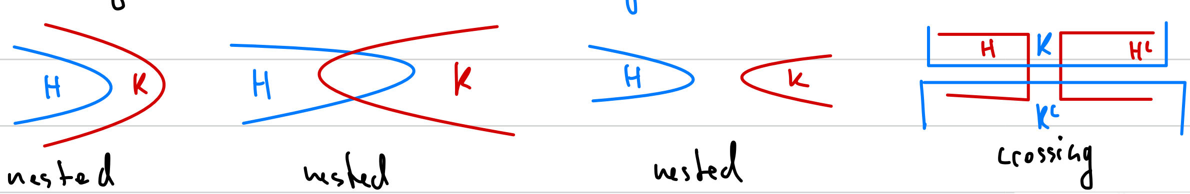
Prop. $M(\mathcal{H})$ is connected. In fact, for any $\mathcal{U}, \mathcal{V} \in \mathcal{U}_0(\mathcal{H})$, $d_{M(\mathcal{H})}(\mathcal{U}, \mathcal{V}) = \frac{1}{2}|\mathcal{U} \Delta \mathcal{V}| < \infty$.

Proof. Since $\mathcal{U} \Delta \mathcal{V}$ is clopen and is a union of isolated points in $\mathcal{H} \setminus \{\emptyset, X\}$, so $\mathcal{U} \Delta \mathcal{U}$ must be finite (otherwise, by the compactness of $\mathcal{H}$, $\mathcal{U} \Delta \mathcal{V}$ would have limit points $\emptyset$ or X).

We now show that $\mathcal{U}$ and $\mathcal{V}$ are connected by a path of length $\frac{1}{2}|\mathcal{U} \Delta \mathcal{V}|$ from $\mathcal{U}$ to $\mathcal{V}$, by induction. Note that $\mathcal{U} \Delta \mathcal{V} = \{H_1, \dots, H_n\} \cup \{H_1^c, \dots, H_n^c\}$, where $H_1, \dots, H_n \in \mathcal{U}$ and $H_1^c, \dots, H_n^c \in \mathcal{V}$. We show that starting with $\mathcal{U}$ and flipping the $H_1, \dots, H_n$ in some order is indeed a path from $\mathcal{U}$ to $\mathcal{V}$.

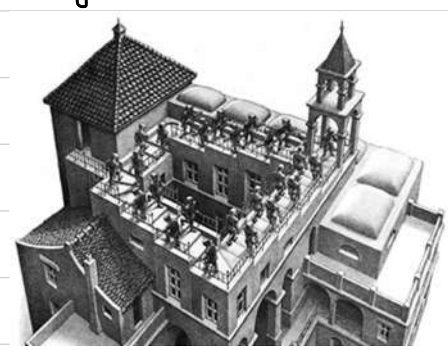
Let H_i be minimal in $\{H_1, \dots, H_n\}$. Then $H_i \in \mathrm{Min}(\mathcal{U})$ because if $K \in \mathcal{U}$ and $K \subseteq H_i$ then $K^c \supseteq H_i^c$ so $K^c \in \mathcal{V}$, so $K \in \{H_1, \dots, H_n\}$, hence $K = H_i$. Thus, $\mathcal{U} — \mathcal{U} \Delta \{H_i, H_i^c\} =: \mathcal{U}_1$ is an edge in $M(\mathcal{H})$ and $\frac{1}{2}|\mathcal{U}_1 \Delta \mathcal{V}| = \frac{1}{2}|\mathcal{U} \Delta \mathcal{V}| - 1$, so by induction, there is a path from $\mathcal{U}_1$ to $\mathcal{V}$. $\square$

What kind of graph is $M(\mathcal{H})$? When is it a tree? We'll first answer the latter question. We say that half-spaces H and K are nested if one of their four corners $H \cap K$, $H^c \cap K$, $H \cap K^c$, $H^c \cap K^c$ is empty; in other words, $H^i \subseteq K^j$ or $H^i \supseteq K^j$ for some $i, j \in \{\pm 1\}$. Otherwise, say that H and K are crossing.



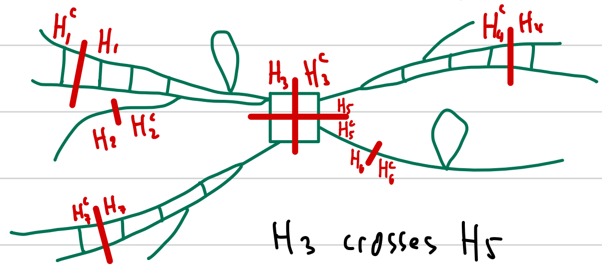
Prop. If $\mathcal{H}$ is nested then $M(\mathcal{H})$ is a tree.

Proof. If $\mathcal{U}_0 \xrightarrow{H_0} \mathcal{U}_1 \xrightarrow{H_1} \mathcal{U}_2 \xrightarrow{H_2} \cdots \xrightarrow{H_{n-1}} \mathcal{U}_n$ is a path in $M(\mathcal{H})$ where $H_i \in \mathcal{U}_i$, then $H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_{n-1}$ because H_i^c and H_{i+1} are minimal in $\mathcal{U}_i$ and nested, so $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_n$ can't be a cycle. $\square$



As pointed out by Marcin Sabok, a counterexample to the argument on the left was provided by Escher in his work "Ascending and Descending" (1960).

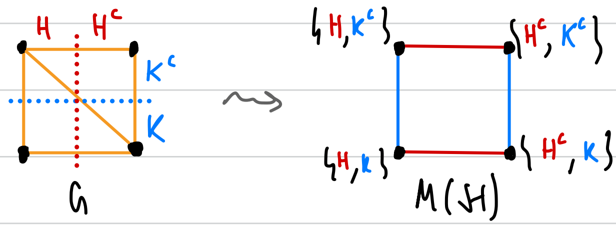
I had proved the last proposition in my "Stallings theorem for CBERs" paper, so I desperately wanted a nested $\mathcal{H}$, but our $\mathcal{H}$ is typically not nested. One could take a maximal nested subset $\mathcal{H}_0 \subseteq \mathcal{H}$, but then typically $M(\mathcal{H}_0)$ is no longer useful as it would not be a standard Borel graph for Borel G and $\mathcal{H}_0$.



H_3 crosses H_5

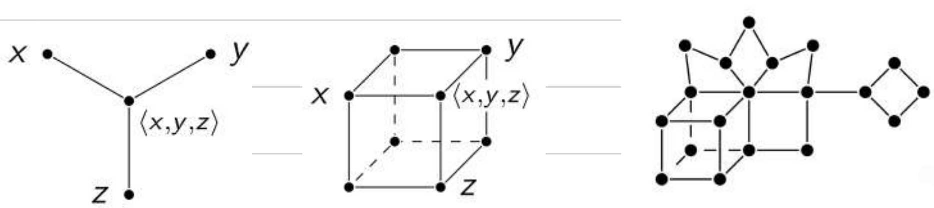
This is when Ronnie pointed out that $M(\mathcal{H})$ had been studied in lattice theory and geometric group theory, and that although it's not a tree, it's a median graph...

Warm-up example.

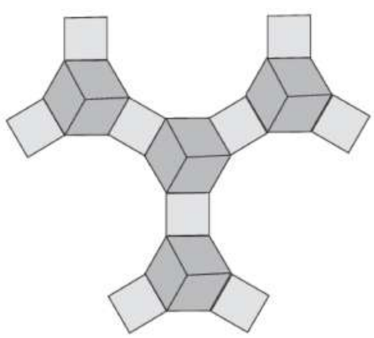


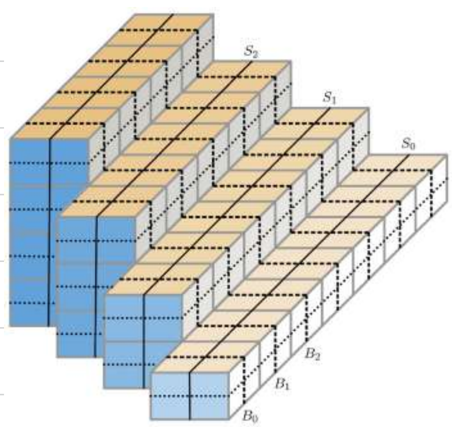
$M(\mathcal{H})$ is "cubic."

A median graph is a graph M where every triple x, y, z of M-connected vertices admits a unique median: a vertex $\langle x, y, z\rangle$ that lies on the intersection of some geodesics P_{xy}, P_{yz}, P_{zx} between $x-y$, $y-z$, and $z-x$, respectively.



Median graphs are exactly the 1-skeleta of CAT(0) cube complexes.





Obs. Trees are median graphs.

Theorem (Isbell '80 – Werner '81, Roller '98). $M(\mathcal{H})$ is a median graph, where for all $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathcal{U}_0(\mathcal{H})$

$$\langle \mathcal{U}, \mathcal{V}, \mathcal{W}\rangle := \{H \in \mathcal{H} : H \text{ is in } \geq 2 \text{ of } \mathcal{U}, \mathcal{V}, \mathcal{W}\} =: P$$
$$= (\mathcal{U} \cap \mathcal{V}) \cup (\mathcal{V} \cap \mathcal{W}) \cup (\mathcal{U} \cap \mathcal{W}).$$

Proof. P is clopen and upward closed by definition, and satisfies the 0/1 law because exactly one of H and H^c wins the majority vote of $\mathcal{U}, \mathcal{V}, \mathcal{W}$. One checks that

$$P \text{ is on a geodesic between } \mathcal{U} \text{ and } \mathcal{V} \iff \mathcal{U} \cap \mathcal{V} \subseteq P \subseteq \mathcal{U} \cup \mathcal{V}, \qquad (*)$$

but the latter is true of P, so P lies on a geodesic from $\mathcal{U}$ to $\mathcal{V}$, similarly other pairs.

Such an ultrafilter P is unique because by $(*)$, P lies on geodesics between every pair $(\mathcal{U},\mathcal{V})$, $(\mathcal{V},\mathcal{W})$, and $(\mathcal{U},\mathcal{W})$ $\iff$ $(\mathcal{U} \cap \mathcal{V}) \cup (\mathcal{V} \cap \mathcal{W}) \cup (\mathcal{U} \cap \mathcal{W}) \subseteq P \subseteq (\mathcal{U} \cup \mathcal{V}) \cap (\mathcal{V} \cup \mathcal{W}) \cap (\mathcal{U} \cap \mathcal{W})$, but both sides are equal, by De Morgan. $\square$

Prop. If $\mathcal{H}$ is (ii) dense towards ends and consists of connected half-spaces, then the median graph $M(\mathcal{H})$ has finite hyperplanes ($= \partial H$ where H is a half-space of M s.t. H and H^c are convex).

The Borelness of $M(G)$ and treeability.

Now let G be a loc. fin. Borel graph with a Borel $\mathcal{H} \subseteq \mathcal{H}(G)$ satisfying (i)-(ii). By applying the structure graph construction to each component of G, we still get a median graph $M(\mathcal{H})$ (with continuum many connected components) on a typically esoteric space $\mathcal{U}_0(\mathcal{H})$ of all clopen ultrafiters on $\mathcal{H}$. This is where the real problem begins! One has to show that (ii) makes $\mathcal{U}_0(\mathcal{H})$ a standard Borel space and $M(G)$ a Borel graph on it. Furthermore, $M(G)$ contains a Borel subtreeing.

Prop. If $\mathcal{H}$ consists of connected half-spaces, then "(ii) being dense towards ends" $\Rightarrow$ $\mathrm{Min}(\mathcal{U})$ is finite for all $\mathcal{U} \in \mathcal{U}_0(\mathcal{H})$. In particular, $M(\mathcal{H})$ is locally finite.

"Proof." Use (ii) to build an open cover of all ends ∂G and take a finite subcover. The connectedness of each $H \in \mathcal{H}$ is used to get that if, for $H, K \in \mathcal{H}$, $\partial H \cap \partial K = \emptyset$, then H and K are nested. $\boxtimes$

Lemma (upgrade to connected). We can take $\mathcal{H}$ to consist of connected half-spaces.

Because the set of finite subsets of a st. Borel space is standard Borel, we get:

Theorem 1. After upgrading $\mathcal{H}$ to consist of connected half-spaces, the space $\mathcal{U}_0(\mathcal{H})$ is standard Borel and $M(\mathcal{H})$ is a locally finite Borel median graph on $\mathcal{U}_0(\mathcal{H})$ with finite hyperplanes. Furthermore, R_G Borel reduces to $R_{M(\mathcal{H})}$, so the treeability of $R_{M(\mathcal{H})}$ would imply that of R_G.

Finally, we came up with a Borel "algorithm" which constructs (bottom-up) a subtreeing of a given Borel median graph with finite hyperplanes, using Lusin-Novikov to partition the collection of all half-spaces into ctbly-many nested Borel families $\mathcal{H}_n$ of half-spaces and building an acyclic subgraph T_n based on $\mathcal{H}_n$ and taking the union $\bigcup_{n \in \mathbb{N}} T_n$. We thus establish the following:

Theorem 2. Let M be a loc. fin. Borel graph with finite hyperplanes. Then M admits a Borel subtreeing $T \subseteq M$, i.e. acyclic Borel subgraph with $R_T = R_M$.