Continuous Model Theory

Lecture 3: Computability and Continuous Model Theory

Bradd Hart

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The meaning of computable in the continuous setting

- In classical logic, when one talks about computable theories or decidable theories, one asks for an algorithm that takes a sentence and correctly returns true or false.
- In the continuous setting we have two problems - we have too many sentences and too many truth values.
- Instead, for a theory T in continuous logic when given some computable set of sentences Σ we ask for an algorithm which, for $\varphi \in \Sigma$ and a computable real $\epsilon > 0$, returns a computable (even rational) interval (a, b) of length at most ϵ such that for all models M of T, $\varphi^M \in (a, b)$. We say that the Σ-theory of T is computable.
- We will mostly concentrate on the case where Σ is some dense set of universal sentences.

Proof systems in continuous logic

- In classical logic, decidability results often go hand in hand with formal proof systems.
- There is a formal proof system for continuous logic but we will not dwell on the details.
- Suffice to say that one starts with a set of assumptions in the form of sentences (which are assumed to have value 0) and then proceeds using deduction rules to conclusions of the form "$r \leq \varphi \leq s$" where $r < s$ are real numbers and φ is a sentence.
- Proofs of this kind are finite and so a proof does not determine the value of a sentence in a theory but only gives you a range.
- We will be most interested in proofs of bounds on universal sentences in the theory of tracial von Neumann algebras of which all matrix algebras are examples as is $\mathcal{R}$.

Connes' Embedding Problem

- The Connes embedding problem (property), CEP, is: Every separable II_1 factor embeds in $\mathcal{R}^{\mathcal{U}}$ where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$.
- Fact: (FHS) The theory of II_1 factors has a computable axiomatization.
- $\mathcal{R} \prec \mathcal{R}^{\mathcal{U}}$ of course but actually any embedding of $\mathcal{R}$ in $\mathcal{R}^{\mathcal{U}}$ is elementary. Moreover, $\mathcal{R}$ embeds into any II_1 factor.
- This means that if CEP holds then the universal theory of $\mathcal{R}$ is determined by the theory of II_1 factors. Why?

The (un)decidability of the universal theory of $\mathcal{R}$

- From below, our matrix argument from the first lecture gives us a lower bound.
- If CEP holds then running proofs from the theory of II_1 factors gives us an upper bound.
- The conclusion is then that if CEP holds then the universal theory of $\mathcal{R}$ is computable.
- Note that the universal theory of $\mathcal{R}$ is computable iff for every computable universal sentence φ and $\epsilon > 0$, there is a computable n_φ such that

$$\left|\varphi^{\mathcal{R}} - \varphi^{M_{n_\varphi}(\mathbb{C})}\right| < \epsilon.$$

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- It is a cooperative game; here are the rules:
- There are two players A(lice) and B(ob). Each are asked one of n "questions" with some distribution (they can be asked different questions).
- They reply with one of k answers - the answers can be chosen probabilistically.
- There is a decision procedure which given the questions and answer pairs, determines if A and B win the game.
- A and B agree ahead of time on a strategy i.e. they can decide on the overall distribution of question and answer pairs given the decision procedure.
- A and B are not allowed to consult each other after they are asked their questions.

What is the value of this game?

- In the end, the chance of winning this game G given distribution μ_A, μ_B on $n \times k$ and a decision function D is

$$\sum_{x,a,y,b} \mu_A(x,a)\mu_B(y,b)D(x,a,y,b)$$

  and we say the value of the game, $v(G)$, is the maximum of this value over all possible distributions.
- These games define the complexity class MIP.
- More formally, L is MIP iff there is an effective map z to games G_z such that $z \in L$ if $v(G_z) \geq 2/3$ and $z \notin L$ if $v(G_z) \leq 1/3$.
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- MIP is a big class: MIP = NEXP.
- With a little wrinkle, it can get bigger.

Non-local quantum games

- We now return to our innocent little game. Once again we are going to ask Alice and Bob to answer some questions and pick answers. Afterwards, we will decide how they did.
- The only twist is that they get access to some quantum entanglement!?! What does this even mean?
- It is a good thing that Alice and Bob are quantum physicists who are well-versed in quantum mechanics, linear algebra and functional analysis!

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- It is a good thing that Alice and Bob are quantum physicists who are well-versed in quantum mechanics, linear algebra and functional analysis!
- First of all, A and B will work together and choose finite-dimensional Hilbert spaces H_A and H_B.
- For each pair of questions x and y, they will choose k-partitions of unity $\{A_a^x : a < k\}$ and $\{B_b^y : b < k\}$ of H_A and H_B respectively.
- A k-partition of unity $\{P_i : i < k\}$ of a Hilbert space H is a set of projections which sum to the identity on H.

Entanglement

- Finally, we entangle things: we also allow A and B to fix a state φ on $H_A \otimes H_B$.
- A state ψ on $B(H)$ is a positive linear functional such that $\psi(id_H) = 1$.
- To be clear then, before A and B are asked their questions x and y, they get to collaborate on the choice of Hilbert spaces, partitions of unity and a choice of state.
- As before, our game has some distribution by which the questions are being asked - call this μ - and there is a decision function D as before.

The value of non-local quantum games

- So for this n-question, k-answer game G and the choices that A and B have made, the expected winning probability is

$$\varphi\left(\sum_{x,y}\mu(x,y)\left(\sum_{a,b}D(x,a,y,b)(A_a^x\otimes B_b^y)\right)\right).$$

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- This quantity is going to be crucial for what we do but I want to note one seeming triviality:

$$\varphi\left(\sum_{x,y}\mu(x,y)\left(\sum_{a,b}D(x,a,y,b)(A_a^x\otimes B_b^y)\right)\right)$$
$$=\sum_{x,y}\mu(x,y)\left(\sum_{a,b}D(x,a,y,b)\varphi(A_a^x\otimes B_b^y)\right).$$

Quantum correlation matrices

- A quantum correlation matrix depends on a set of data:
  - a choice of two finite-dimensional Hilbert spaces H_A and H_B,
  - a number n such that for every $x, y < n$, we have k-partitions of unity
$$\{A^x_a : a < k\} \text{ and} \{B^y_b : b < k\}$$
of H_A and H_B respectively, and
  - a choice of state φ on $H_A \otimes H_B$.
- With this data, we let $p_{xayb} = \varphi(A^x_a \otimes B^y_b)$ for all $x, y < n$ and $a, b < k$.
- A quantum correlation matrix p is of the form (p_{xayb}) for some set of data as above and we let $C_q(n, k)$ be the set of all quantum correlation matrices for fixed n and k.

MIP*

- We can then write

$$v^*(G) = \sup_{p \in C_q(n,k)} \sum_{x,y} \mu(x,y) \left(\sum_{a,b} D(x,a,y,b) p_{xaby} \right).$$

- We define the complexity class MIP* by L is in MIP* iff there is an effective map z to games G_z such that $z \in L$ if $v^*(G_z) \geq 2/3$ and $z \notin L$ if $v^*(G_z) \leq 1/3$.

The Tsirelson problem

- We record another possible approach to quantum correlation matrices due to Tsirelson.
- Again, we fix n and k but now we work on a single separable Hilbert space H.
- We fix projections A_a^x and B_b^y on H such that for each x and y, $\{A_a^x : a < k\}$ and $\{B_b^y : b < k\}$ are partitions of unity.
- Moreover A_a^x and B_b^y commute for every x, a, y, b. This is a generalization of the tensor product scenario.
- If we pick a state ψ on H, we let $p_{xayb} = \psi(A_a^x B_b^y)$ for all $x, y < n$ and $a, b < k$.
- We let $C_{qc}(n, k)$ be the set of all quantum correlation matrices obtained in this manner.
- The Tsirelson problem is whether $\overline{C_q(n, k)} = C_{qc}(n, k)$ for all n and k.

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- It is highly non-trivial but MIP* = RE! This was first announced in 2019 by Ji, Natarjan, Vidick, Wright and Yuen.The proof is *extremely* interesting.
- MIP* is contained in RE so it suffices to show that the Halting problem is in MIP*.
- They show that for any Turing machine M, there is an effective assignment of a game G_M such that $v^*(G_M) = 1$ if M halts and $v^*(G_M) \leq 1/2$ if M does not halt. This is enough to get MIP* = RE.
- They do more than that though!

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- MIP*=RE implies the Tsirelson problem does not hold.
- CEP holds iff there is a positive resolution of the Tsirelson problem. (Ozawa; Fritz and Junge et al.; Ozawa again)
- So CEP is false!
- In fact, one can express the value of a non-local quantum game using a universal sentence in continuous model theory.
- So if you could compute approximate the value of universal sentences in $\mathcal{R}$ then you would be able to solve the halting problem hence the universal theory of $\mathcal{R}$ is not computable.