Continuous Model Theory

Lecture 2: Ultraproducts and Metric Structures

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May 6 - 8, 2026

Filters and Ultrafilters

Definition

If X is a set and $F \subseteq \mathcal{P}(X)$ then F is said to be a filter if

- $\emptyset \notin F$,
- if $A, B \in F$ then $A \cap B \in F$, and
- if $A \in F$ and $A \subseteq B \subseteq X$ then $B \in F$.

Lemma

$G \subseteq \mathcal{P}(X)$ is contained in a filter iff G has the finite intersection property i.e. for every finite $G_0 \subseteq G$, $\bigcap G_0 \neq \emptyset$.

Definition

An ultrafilter on X is a filter F such that for every $A \subseteq X$, either $A \in F$ or $X \setminus A \in F$.

Lemma

- *If F is a filter on X then F is an ultrafilter iff it is a maximal filter.*
- *Any filter on X can be extended to an ultrafilter.*

Ultralimits

Now suppose $\mathcal{U}$ is an ultrafilter on a set I and $\bar{r} = \langle r_i : i \in I \rangle$ is an I-indexed family of real numbers. We define the ultralimit of $\bar{r}$ with respect to $\mathcal{U}$ as follows:

$$\lim_{i \to \mathcal{U}} r_i = r \text{ iff for every } \epsilon > 0, \{i \in I : |r - r_i| < \epsilon\} \in \mathcal{U}.$$

Lemma

If $\bar{r}$ is bounded then

- $\lim_{i \to \mathcal{U}} r_i$ *exists and is unique.*
- $\lim_{i \to \mathcal{U}} r_i = \inf\{B : \{i \in I : r_i < B\} \in \mathcal{U}\}$.
- $\lim_{i \to \mathcal{U}} r_i = \sup\{B : \{i \in I : r_i > B\} \in \mathcal{U}\}$.

Ultraproducts of metric spaces

Fix an index set I, an ultrafilter $\mathcal{U}$ on I and uniformly bounded metric spaces (X_i, d_i) for $i \in I$ i.e. there is some B so that for all i and all $x, y \in X_i$, $d_i(x, y) \leq B$. Define d on $\prod_{i \in I} X_i$ as follows:

$$d(\overline{x}, \overline{y}) = \lim_{i \to \mathcal{U}} d_i(x_i, y_i)$$

Lemma

d is a pseudo-metric on $\prod_{i \in I} X_i$.

Definition

The ultraproduct of the X_i's with respect to $\mathcal{U}$, $\prod_{\mathcal{U}} X_i$, is the metric space obtained by quotienting $\prod_{i \in I} X_i$ by d. If all the X_i's are equal to a fixed X we will often write $X^{\mathcal{U}}$ for this ultraproduct and call it the ultrapower.

Exercises

- Show that for any I and ultrafilter $\mathcal{U}$ on I, $[0,1]^{\mathcal{U}} \cong [0,1]$. More generally, show that for a compact metric space X, $X^{\mathcal{U}} \cong X$.
- Show that if X_i is an I-indexed family of uniformly bounded complete metric spaces then $\prod_{\mathcal{U}} X_i$ is complete.
- Show for any family of uniformly bounded metric spaces X_n for $n \in \mathbb{N}$, $\prod_{\mathcal{U}} X_n$ is complete if $\mathcal{U}$ is non-principal.
- Show that this definition of ultraproduct is the same as the discrete or set-theoretic ultraproduct when the metric on the metric spaces is discrete $\{0,1\}$-valued.

Metric structures

- We want to add more structure to a (bounded) metric space; for now let's consider a single additional function f.
- So we will have a bounded metric space (X,d) and a function f say of one variable. We do want that the ultraproduct of these structures is still a structure of the same kind. So how do we define f on the ultrapower of X?
- f must be continuous!
- f must be uniformly continuous!
- There is nothing special about one variable; these arguments apply to functions of many variables.

Metric structures cont'd

- What about relations? Imagine that we have a one-variable relation R (taking values somewhere) on a metric space and we want to make sense of it in the ultrapower.
- Its codomain must be compact and R must be uniformly continuous.
- There is really no loss in assume that the codomain of R is $[0,1]$ or some other compact interval in the reals.
- Again there is nothing special about one-variable; we can have relations of many variables.

The language of a metric structure

A language $\mathcal{L}$ will consist of

- a set $\mathcal{S}$ called sorts;
- $\mathcal{F}$, a family of function symbols. For each $f \in \mathcal{F}$ we specify the domain and codomain of f: $\text{dom}(f) = \prod_{i=1}^{n} S_i$ where $S_1, \ldots, S_n \in \mathcal{S}$ and $\text{cod}(f) = S$ where $S \in \mathcal{S}$. Moreover, we also specify a continuity modulus. That is, for each $i \leq n$ we are given $\delta_i^f : [0, 1] \to [0, 1]$; and
- $\mathcal{R}$, a family of relation symbols. For each $R \in \mathcal{R}$ we are given the domain $\text{dom}(R) = \prod_{i=1}^{n} S_i$ where $S_1, \ldots, S_n \in \mathcal{S}$ and the $\text{cod}(R) = K_R$ for some closed interval K_R. Moreover, for each i, we specify a continuity modulus $\delta_i^R : [0, 1] \to [0, 1]$.
- For each $S \in \mathcal{S}$, we have one special relation symbol d_S with domain $S \times S$ and codomain of the form $[0, B_s]$. Its continuity moduli are the identity functions.

Definition of a metric structure

A metric structure $\mathcal{M}$ *interprets* a language $\mathcal{L}$; it will consist of

- an $\mathcal{S}$-indexed family of complete bounded metric spaces $(S^{\mathcal{M}}, d_S^{\mathcal{M}})$ with bound B_S for $S \in \mathcal{S}$;
- a family of functions $f^{\mathcal{M}}$ for every $f \in \mathcal{F}$ such that $\mathrm{dom}(f^{\mathcal{M}}) = \prod_{i=1}^n S_i^{\mathcal{M}}$ where $\mathrm{dom}(f) = \prod_{i=1}^n S_i$ and $\mathrm{cod}(f^{\mathcal{M}}) = S^{\mathcal{M}}$ where $\mathrm{cod}(f) = S$. $f^{\mathcal{M}}$ is uniformly continuous as specified by the uniform continuity moduli associated to f; that is for every $i \leq n$, $a_1 \in S_i^{\mathcal{M}}, \ldots, a_n \in S_n^{\mathcal{M}}$ and $a_i' \in S_i^{\mathcal{M}}$, for every $\epsilon > 0$ and $\delta = \delta_i^f(\epsilon)$ if $d_{S_i}^{\mathcal{M}}(a_i, a_i') < \delta$ then

$$d_S^{\mathcal{M}}(f^{\mathcal{M}}(a_1, \ldots, a_i, \ldots, a_n), f^{\mathcal{M}}(a_1, \ldots, a_i', \ldots, a_n)) < \epsilon;$$

- a family of relations $R^{\mathcal{M}}$ for every $R \in \mathcal{R}$ such that $\mathrm{dom}(R^{\mathcal{M}}) = \prod_{i=1}^n S_i^{\mathcal{M}}$ where $\mathrm{dom}(R) = \prod_{i=1}^n S_i$ and $\mathrm{cod}(R^{\mathcal{M}}) = K_R$. $R^{\mathcal{M}}$ is uniformly continuous as specified by the uniform continuity moduli associated to R as above.

Examples of metric structures

Some simple examples:

- Any complete bounded metric space (X, d). This has the empty family of functions and relations although we often count the metric as a relation (why is it uniformly continuous?)
- Any ordinary first order structure M with some collection of functions and relations. To see this as a metric structure, we put the discrete $\{0, 1\}$-valued metric on M to make it a bounded metric space. All functions become uniformly continuous. Relations which are usually thought of as subsets of M^n become $\{0, 1\}$-valued functions - again they are uniformly continuous.

Hilbert space

- A Hilbert space H is a complete complex inner product space; how can we see this as a metric structure?
- Let B_n be the ball of radius n centered at the origin in H; B_n is a bounded complete metric space with respect to the metric induced by the inner product.
- There are inclusion maps between B_n and B_m if $n \leq m$.
- 0 is a constant (our functions can have arity 0!) in B_1.
- For complex numbers λ and for every n, there is a unary function λ_n which is scalar multiplication by λ on B_n; this function has range in B_m where m is the least integer greater than or equal to $n|\lambda|$.
- The operation of addition has to be similarly divided up: for $m, n \in \mathbb{N}$, there is an operation $+_{m,n}$ which takes $B_m \times B_n$ to B_{m+n}.

Hilbert space, cont'd

- The inner product is complex valued which is an additional issue. Besides dividing it up so that there is a relation defined on each product $B_m \times B_n$, we also have to separate this relation into its real and complex parts.
- So formally a Hilbert space can be thought of as a metric structure by considering
  - The family of bounded metric structures B_n for all $n \in \mathbb{N}$;
  - the family of functions 0, λ_n for $\lambda \in C$ and $n \in \mathbb{N}$, inclusion maps between the sorts, and $+_{m,n}$ for all $m, n \in \mathbb{N}$; and
  - the family of relations $Re(\langle -, - \rangle)_{m,n}$ and $Im(\langle -, - \rangle)_{m,n}$ for $m, n \in \mathbb{N}$.
- One checks easily that these functions and relations are uniformly continuous. The relations have bounded range since they are restricted to bounded balls.

Ultraproducts of metric structures

Fix a language $\mathcal{L}$, an index set I, an ultrafilter $\mathcal{U}$ on I and $\mathcal{L}$-structures $\mathcal{M}_i$ for $i \in I$.

Definition

The ultraproduct of the $\mathcal{M}_i$'s with respect to $\mathcal{U}$, $\prod_{\mathcal{U}} \mathcal{M}_i$ is the $\mathcal{L}$-structure $\mathcal{M}$ defined as follows:

1. for every sort S, $S^{\mathcal{M}} = \prod_{\mathcal{U}} S_{\mathcal{M}_i}$,
2. for every function symbol f
$$f^{\mathcal{M}} = (f^{\mathcal{M}_i} : i \in I)/\mathcal{U}, \text{ and}$$
3. for every relation symbol R,
$$R^{\mathcal{M}} = \lim_{i \to \mathcal{U}} R^{\mathcal{M}_i}.$$

If all of the $\mathcal{M}_i$'s are a fixed $\mathcal{N}$, we call this the ultrapower and write $\mathcal{N}^{\mathcal{U}}$.

Terms and formulas

For a language $\mathcal{L}$, terms are defined inductively from function symbols and variables by composition exactly as in discrete logic. The only wrinkle is that one needs to keep track of the continuity modulus of terms determined by composition. Formulas are defined inductively.

Definition

- Suppose R is a relation symbol in $\mathcal{L}$ with $\mathrm{dom}(R) = \prod_{i=1}^{n} S_i$ and $\mathrm{cod}(R) = K_R$, and τ_i are terms where $\mathrm{cod}(\tau_i) = S_i$ for all i. Then $R(\tau_1, \ldots, \tau_n)$ is a formula. The domain, codomain and continuity moduli are those obtained by composition.
- Suppose $\varphi_i(\overline{x})$ is a formula with codomain K_{φ_i} for all $i \leq n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Then $f(\varphi_1, \ldots, \varphi_n)$ is a formula with codomain $f(\prod_{i=1}^{n} K_{\varphi_i})$ and domain and continuity moduli determined by composition.
- If φ is a formula and x is a variable then $\sup_x \varphi$ and $\inf_x \varphi$ are both formulas. The sort of x is removed from the domain; the codomain and continuity moduli for the remaining variables stay the same.

Interpretations

Fix a metric structure $\mathcal{M}$ for a language $\mathcal{L}$.

- Terms are interpreted by composition inductively as usual.
- For the formula $R(\tau_1(\overline{x}), \ldots, \tau_n(\overline{x}))$ where R is a relation in $\mathcal{L}$ and $\tau_1, \ldots, \tau_n$ are terms, its interpretation is given, for every appropriate $\overline{a} \in \mathcal{M}$, by

$$R^M(\tau_1^{\mathcal{M}}(\overline{a}), \ldots, \tau_n^{\mathcal{M}}(\overline{a}))$$

- If $\varphi_i(\overline{x})$ is a formula for all $i \leq n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function then if ψ is the formula $f(\varphi_1, \ldots, \varphi_n)$ then $\psi^M := f(\varphi_1^{\mathcal{M}}, \ldots, \varphi_n^{\mathcal{M}})$.
- Suppose $\varphi(x, \overline{y})$ is a formula and $\overline{a} \in \mathcal{M}$ is a tuple appropriate for the variables $\overline{y}$ and x is of sort S. Then

$$(\sup_x \varphi(x, \overline{a}))^{\mathcal{M}} := \sup\{\varphi^{\mathcal{M}}(b, \overline{a}) : b \in S^{\mathcal{M}}\}$$

and

$$(\inf_x \varphi(x, \overline{a}))^{\mathcal{M}} := \inf\{\varphi^{\mathcal{M}}(b, \overline{a}) : b \in S^{\mathcal{M}}\}.$$

Basic properties

Proposition

In an $\mathcal{L}$-structure $\mathcal{M}$

- *the interpretations of terms in $\mathcal{M}$ are uniformly continuous functions with continuity moduli specified by the definition of the term, and*
- *all formulas when interpreted in $\mathcal{M}$, define uniformly continuous functions with domains, codomains and continuity moduli specified by the definition.*

A sentence is a formula with no free variables. It is a consequence of the Proposition that any sentence in $\mathcal{L}$ takes on a value in a metric structure in a compact interval specified by $\mathcal{L}$ and this interval is independent of the given structure.

Theories

- For a language $\mathcal{L}$, $\text{Sent}_{\mathcal{L}}$ is the set of sentences of $\mathcal{L}$.
- The theory of an $\mathcal{L}$-structure $\mathcal{M}$ is the function $\text{Th}(\mathcal{M}) : \text{Sent}_{\mathcal{L}} \to \mathbb{R}$ defined by, for any sentence φ,

  $$\text{Th}(\mathcal{M})(\varphi) = \varphi^{\mathcal{M}}$$

  Notice that $\text{Th}(\mathcal{M})$ is a linear functional on the space of sentences and is in fact determined by its kernel. We then sometimes refer to $\{\varphi \in \text{Sent}_{\mathcal{L}} : \varphi^{\mathcal{M}} = 0\}$ as the theory of $\mathcal{M}$.
- An ($\mathcal{L}$-)theory is a set of sentences T which is contained in $\text{Th}(\mathcal{M})$ for some $\mathcal{M}$.

Example

We will write out and interpret some formulas and sentences about Hilbert spaces.

- There are universal (sup) sentences expressing the fact that we have a complex inner product space. For instance, we have

  $$\sup_{x \in B_1} \sup_{y \in B_1} d_{B_2}(x +_{1,1} y, y +_{1,1} x)$$

  which evaluates to 0 and partially expresses that + is commutative.
- We have the relationship between the inner product and the metric:

  $$\sup_{x \in B_n} \sup_{y \in B_n} (d_{B_n}(x, y)^2 - re(\langle x - y, x - y \rangle)).$$
- We also have $\sup_{x \in B_1}(d(x, 0) \dot{-} 1)$.

Example, cont'd

- Consider the sentence, for every $n \in \mathbb{N}$,

$$\sup_{x \in B_n} \min\{1 \dot{-} d(x,0), \inf_{y \in B_1} d(x, i(y))\}$$

- There are sentences expressing that a Hilbert space is infinite-dimensional (exercise).
- All the sentences we have written out or alluded to specify the theory of infinite-dimensional Hilbert spaces.
- Not too surprisingly it has exactly one separable model up to isomorphism - this is called being separably categorical.
- By an argument which only makes sense later this lecture but very similar to the discrete case, these sentences determine the theory of any model i.e. the theory is complete.

Łoś' Theorem

Theorem

Suppose $\mathcal{M}_i$ are $\mathcal{L}$-structures for all $i \in I, \mathcal{U}$ is an ultrafilter on I, $\varphi(\overline{x})$ is an $\mathcal{L}$-formula and $\overline{a} \in \mathcal{M} := \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ then

$$\varphi^{\mathcal{M}}(\overline{a}) = \lim_{i \to U} \varphi^{\mathcal{M}_i}(\overline{a}_i).$$

Satisfiability

Definition

- We say a set of sentences Σ in a language $\mathcal{L}$ is satisfied if there is an $\mathcal{L}$-structure $\mathcal{M}$ such that for every sentence in Σ holds in $\mathcal{M}$ i.e. for every $\varphi \in \Sigma$, $\varphi^{\mathcal{M}} = 0$.
- We say such a Σ is finitely satisfied if every finite subset of Σ is satisfied.
- For a set of sentence Σ and $\epsilon > 0$, the ϵ-approximation of Σ is

$$\{|\varphi| \dot{-} \epsilon : \varphi \in \Sigma\}$$

- Σ is approximately finitely satisfied if for every $\epsilon > 0$, the ϵ-approximation of Σ is finitely satisfiable.

Compactness

Theorem

TFAE for a set of sentences Σ in a language $\mathcal{L}$

- Σ *is satisfiable.*
- Σ *is finitely satisfiable.*
- Σ *is approximately finitely satisfiable.*

Embeddings and elementary submodels

- Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures such that the universe of $\mathcal{M}$ is a closed subset of $\mathcal{N}$. $\mathcal{M}$ is called a submodel if all functions and relations from $\mathcal{L}$ on $\mathcal{M}$ are the restriction of those from $\mathcal{N}$. We write $\mathcal{M} \subseteq \mathcal{N}$.
- For $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M}$ is an *elementary* submodel if, for every $\mathcal{L}$-formula $\varphi(\overline{x})$ and every $\overline{a} \in \mathcal{M}$, $\varphi^{\mathcal{M}}(\overline{a}) = \varphi^{\mathcal{N}}(\overline{a})$. We write $\mathcal{M} \prec \mathcal{N}$.
- An embedding between metric structures is a map which preserves the functions and relations. An embedding is elementary if its image is an elementary submodel of the codomain.
- For a theory T, Mod(T) is the category of models of T with elementary maps as morphisms. Such a class is called an elementary class.

Notice that by Łoś' Theorem, any metric structure $\mathcal{M}$ embeds elementarily into its ultrapower $\mathcal{M}^{\mathcal{U}}$ for any ultrafilter $\mathcal{U}$ via the diagonal embedding.

Downward Löwenheim-Skolem

Proposition (Tarski-Vaught)

If $\mathcal{M} \subseteq \mathcal{N}$ then $\mathcal{M}$ is an elementary submodel if for every formula $\varphi(x, \overline{y})$, $r \in \mathbb{R}$ and $\overline{a} \in \mathcal{M}$, if $(\inf_x \varphi(x, \overline{a}))^{\mathcal{N}} < r$ then there is $b \in \mathcal{M}$ such that $(\varphi(b, \overline{a}))^{\mathcal{N}} < r$.

Theorem (DLS)

Suppose that $\mathcal{N}$ is an $\mathcal{L}$-structure and A is a subset of $\mathcal{N}$. Then there is an elementary submodel $\mathcal{M} \subseteq \mathcal{N}$ such that

1. *A is contained in $\mathcal{M}$ and*
2. *for every sort S,*

$$\chi(S^{\mathcal{M}}) \leq \chi(\mathcal{L}) + \chi(A)$$

Some abstract model theory

Theorem

For a class of $\mathcal{L}$-structures $\mathcal{C}$, TFAE

1. *$\mathcal{C}$ is an elementary class.*
2. *$\mathcal{C}$ is closed under isomorphisms, ultraproducts and elementary submodels.*
3. *$\mathcal{C}$ is closed under isomorphisms, ultraproducts and ultraroots.*

Theorem

Continuous first order logic is the maximal logic on metric structures which satisfies compactness, the downward Löwenheim-Skolem theorem and unions of elementary chains.