Continuous Model Theory and Computability

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Overview

- Lecture 1: An example
- Lecture 2: A crash course in continuous model theory
- Lecture 3: Now let's work some computability in

Let's start with an example: matrices and norms

- Fix a matrix A, say a complex $n \times n$ matrix; we write $A \in M_n(\mathbb{C})$.
- We want to concentrate on two norms; the easiest is the operator norm:

$$\|A\| = \sup_{x \in \mathbb{C}^n, x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

  This is the norm associated to A as a linear operator on $\mathbb{C}^n$ and is always bounded (the linear operator is continuous).
- We can compute $\|A\|$ in a different way. Consider $A^* A$ which is a semi-positive, self-adjoint (hermitian) matrix. It is unitarily diagonalizable so

$$\|A\| = \sqrt{\|A^* A\|} = \sqrt{\lambda}$$

  where λ is the largest eigenvalue of $A^* A$.

Norms continued

- The second norm goes by different names: the 2-norm, the Hilbert-Schmidt norm or the normalized Euclidean norm.
- Identify $M_n(\mathbb{C})$ with $\mathbb{C}^{n^2}$ and take the usual Euclidean norm, normalized by $\frac{1}{\sqrt{n}}$.
- One can write this as

$$\|A\|_2 = \sqrt{\frac{Tr(A^*A)}{n}}$$

  where Tr is the usual matrix trace. We will write τ_n for $\frac{Tr}{n}$.
- You should convince yourself that if $A = (a_{ij})$ then

$$Tr(A^*A) = \sum_{i,j} \|a_{ij}\|^2.$$

- Why do we normalize? We want $\|I_n\|_2 = 1$ for all n.

Topologies

- For any fixed n, the operator norm and the 2-norm yield the same topology on $M_n(\mathbb{C})$.
- In fact, for any $A \in M_n(\mathbb{C})$, if λ is the largest eigenvalue of A^*A then

$$\lambda \leq Tr(A^*A) \leq n\lambda$$

and so

$$\frac{\|A\|}{\sqrt{n}} \leq \|A\|_2 \leq \|A\|$$

which means that any operator norm ball contains a 2-norm ball and vice versa if we fix n.
- So of course we are not going to fix n.

Embeddings of Matrix algebras

- Suppose k divides n and we look at the map $i_k^n \colon M_k(\mathbb{C}) \hookrightarrow M_n(\mathbb{C})$ given by

$$A \mapsto \begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & & \\ \vdots & & \ddots & \\ 0 & & & A \end{pmatrix}$$

  where there are m blocks of A on the diagonal and $km = n$.
- What type of map is this? It is a *-homomorphism: it preserves $+, \cdot, *$, the identity.
- What about the norms? It is clear that the normalized trace is preserved. So is the operator norm - think about the largest eigenvalue of $A^* A$.

Why do we care about these embeddings?

- It is a fact that if $j\colon M_k(\mathbb{C}) \hookrightarrow M_n(\mathbb{C})$ is a unital *-homomorphism then there is a unitary $U \in M_n(\mathbb{C})$ such that $U^* j U = i_k^n$. (JvN, 1942)
- In particular, $M_k(\mathbb{C}) \hookrightarrow M_n(\mathbb{C})$ via a unital *-homomorphism iff k divides n.

Some (abstract) calculations in matrix algebras

- We start with the case of the 2-norm. Consider expressions of the form

$$\varphi(x_1, \dots, x_l) = f(\|p_1(x_1, \dots, x_l)\|_2, \dots, \|p_m(x_1, \dots, x_l)\|_2)$$

  where $p_1, \dots, p_m$ are *-polynomials over $\mathbb{C}$ in non-commuting variables and f is a continuous function from $\mathbb{R}^m$ to $\mathbb{R}$.
- If we have $a_1, \dots a_l \in M_n(\mathbb{C})$ then we compute the value of $\varphi(a_1, \dots, a_l)$ and it yields a real number.
- Now I wish to do the following calculation: let $\bar{a} = a_1, a_2, \dots, a_l$ range over the operator norm unit ball, $B_1(M_n(\mathbb{C}))$ and compute

$$s_n = \sup_{\bar{a} \in B_1(M_n(\mathbb{C}))} \varphi(\bar{a}).$$

- Notice that if k divides n then $s_k \leq s_n$.

The calculation continued

- Notice also that there is an absolute upper bound on s_n based only on φ and independent of n.
- So we have a directed system based on divisibility and we can conclude that

  $$s_\varphi = \lim_{n\to\infty} s_n$$

  exists. What is this the value of and where is it being computed?
- Divisibility also gives us a directed system of embeddings among matrix algebras and we can take direct limit and call it M_∞; maybe concretely we can think of M_∞ as the union of the following chain:

  $$M_2(\mathbb{C}) \hookrightarrow M_{3!}(\mathbb{C}) \hookrightarrow M_{4!}(\mathbb{C}) \hookrightarrow \ldots M_\infty.$$

Topology on M_∞

- We add one more thing: there is a topology induced by the 2-norm on M_∞ and this topology is not complete.
- We can actually see this pretty easily - consider projections π in $M_n(\mathbb{C})$; they will satisfy

$$\|\pi\|_2 = \tau_n(\pi) = \frac{k}{n}$$

  for some $0 \leq k \leq n$. So M_∞ only contains projections with rational 2-norms.
- It is easy to construct a sequence of projections in M_∞ which forms a Cauchy sequence with respect to the 2-norm but the limit of the trace on this sequence is not rational (exercise).

The magical $\mathcal{R}$

- We now take the completion of M_∞ with respect to the 2-norm (on operator norm bounded sequences) and we call this $\mathcal{R}$. It really should have a better name but it is called the hyperfinite II_1 factor - it is a fundamental object in the study of von Neumann algebras.
- Since the operations of $+, \cdot, *$ are continuous with respect to the 2-norm, these operations are also defined on $\mathcal{R}$.
- Of course there is also an identity element - the limit of the identity elements from the matrix algebras. So it is a complex *-algebra with a unit and a trace - $\mathcal{R}$ is complete with respect to the 2-norm associated to the trace on operator norm balls.
- We now have our answer regarding s_φ: the limit s_φ satisfies

$$s_\varphi = \sup_{\bar{a} \in B_1(\mathcal{R})} \varphi(\bar{a}).$$

What about the operator norm?

- Everything we said about the 2-norm could be repeated for the operator norm.
- Consider an expression of the form

$$\varphi(x_1, \ldots, x_l) = f(\|p_1(x_1, \ldots, x_l)\|, \ldots, \|p_m(x_1, \ldots, x_l)\|)$$

  where $p_1, \ldots, p_m$ are *-polynomials over $\mathbb{C}$ in non-commuting variables and f is a continuous function from $\mathbb{R}^m$ to $\mathbb{R}$.
- We again want to do the following calculation: let $\bar{a} = a_1, a_2, \ldots, a_l$ range over the operator norm unit ball, $B_1(M_n(\mathbb{C}))$ and compute

$$s_n = \sup_{\bar{a} \in B_1(M_n(\mathbb{C}))} \varphi(\bar{a}).$$

- Once again if k divides n then $s_k \leq s_n$.
- We also argue again that the s_n are uniformly bounded independent of n.
- So we once again have $s_\varphi = \lim_{n\to\infty} s_n$ and we wonder in what algebra does this value have a meaning.

The exciting $\mathcal{Q}$

- We will do this quickly since you probably see the pattern. We start again with M_∞ and now we notice that it also carries a topology arising from the operator norm.
- Again this topology is not complete although projections will not help us this time (exercise: why?) but we can take the completion of M_∞ with respect to the operator norm and we get the algebra $\mathcal{Q}$ - the universal uniformly hyperfinite (UHF) algebra.
- All the algebraic operations are continuous with respect to the operator norm and so once again $\mathcal{Q}$ has a natural unital *-algebra structure.
- Interestingly it also has a trace which is continuous with respect to the operator norm but $\mathcal{Q}$ is not complete with respect to the associated 2-norm.
- s_φ turns out to be the value of $\sup_{\bar{a} \in B_1(\mathcal{Q})} \varphi(\bar{a})$.

Direct limits of matrix algebras, again!

- Let's look at two direct limits of matrix algebras:

$$M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C}) \hookrightarrow M_8(\mathbb{C}) \hookrightarrow \dots M_{2^\infty}$$

and

$$M_3(\mathbb{C}) \hookrightarrow M_9(\mathbb{C}) \hookrightarrow M_{27}(\mathbb{C}) \hookrightarrow \dots M_{3^\infty}$$

- I would like to argue that these two algebras, when one considers them with the operator norm, are NOT the same.
- We see this with partitions of unity: we are asking if 1 (the identity) can be divided into n equal pieces.
- Maybe more precisely, are there orthogonal, conjugate projections $\pi_1, \pi_2, \dots, \pi_n$ such that $\sum \pi_j = 1$.
- Notice that the answer is yes for $n = 2^k$ for all k in M_{2^∞} and the answer is yes for $n = 3^k$ for all k in M_{3^∞}.
- But the answer is no for $n = 2$ in M_{3^∞}; why?

Amazing fact

- If you complete M_{2^∞} and M_{3^∞} with respect to the 2-norm then they ARE isomorphic and they are isomorphic to $\mathcal{R}$.
- This is not at all obvious but let's understand why when we focus on the 2-norm, we can get Cauchy sequence in M_{3^∞} that acts like a partition of unity for $n = 2$.

Approximating the value of s_φ

- Suppose we wanted to compute the value of s_φ in $\mathcal{R}$; recall this is

$$\lim_{n\to\infty} \sup_{\bar{a}\in B_1(M_n(\mathbb{C}))} \varphi(\bar{a})$$

  where φ is a continuous function applied to terms of the form $\|p(\bar{a})\|_2$ for some *-polynomial p.
- Maybe we should say why we might want to do this.
- Originally, since we knew that this limit existed, some of us were interested in approximating these values, obtaining estimates on what sizes of matrices you need to approximate the value to within ϵ - the usual kind of mathy stuff that we do.
- It turns out, for well chosen φ, s_φ is the winning probability for certain non-local quantum games. We will talk more about this in the third lecture but as we will see - this is important.

A procedure to obtain a lower bound

- We go back and look at the operator norm unit ball of $M_n(\mathbb{C})$.
- I hope it is believable that we can compute, for any δ, a finite list of complex-rational valued $n \times n$ matrices which are δ-dense in the unit ball.
- Since φ is a continuous function of the inputs, for any $\epsilon > 0$ there is a $\delta > 0$ such that we can create a δ-dense list and compute an ϵ-approximation of s_n.
- By choosing ϵ smaller as n tends to infinity, we can create an increasing sequence of s_n's and an improving lower bound for s_φ.

Some foreshadowing of computability

- There are a few issues we will have to sort out.
- One small computability issue here is that for a given φ we will need to know computably how to obtain δ from ϵ.
- In fact, we can't use all φ - there are too many - but since we are only interested at the moment in approximating the value of s_φ, it suffices to have some suitably dense collection of expressions φ. More on this as we go along.
- Big surprise: we cannot do this at all! We will see in the final lecture that there is no algorithm that when given φ and $\epsilon > 0$ will be able to tell you which n will guarantee that s_n is within ϵ of s_φ.