An annotated bibliography for the ISM short course on continuous model theory and computability, May 6 - 8, 2026, Bradd Hart

1 The origins of continuous model theory

1. Lukasiewicz, J. and Tarski, A., Untersuchungen über den Aussagenkalkül, Comptes rendus de Varsovie, classe III, 23, 30 – 50.

   I put this here just to point out that the idea of a multi-valued, even infinitely valued, logic (at least a propositional logic) is a very old one.

2. Chang, C.C. and Keisler, H.J., Continuous Model Theory, Annals of Math. Studies, 58, 1966.

   Chang and Keisler introduced a sweeping notion of model theory which would use continuous connectives and would take truth values in compact spaces. They defined ultraproducts and proved basic abstract notions for this model theory but the syntax was unwieldy and it never caught on.

3. Henson, C. W. Nonstandard hulls of Banach spaces. IJM, 25, 108–144, 1976.

   Henson introduced the notion of positive bounded logic which acted as the logic for Banach space based structures. It was a precursor to metric logic and many of the foundational aspects of the theory of metric structures derive their origins from this work on positive bounded logic.

4. Henson, C. W., and Iovino, J., Ultraproducts in analysis. In C. Finet and C. Michaux (Eds.), Analysis and logic, 1–110, 2003, Cambridge University Press.

   This paper is interesting because it was published almost coincident with the development of CATS and modern continuous model theory. It studies the ultraproduct construction and its uses in analysis in the style of positive bounded logic but in retrospect was already doing continuous model theory.

5. Hart, B., Kim, B. and Pillay, A., Coordinatisation in simple theories, JSL, 65 , 293–309, 2000.

   The notion of hyperimaginaries is introduced in this paper and in retrospect, this can be seen as the start of continuous logic in its modern form. Hyperimaginaries are quotients by type-definable equivalence relations and this paper contains an adhoc logic which can be used to capture them. Continuous model theory captures hyperimaginaries far more cleanly.

6. Ben Yaacov, I., Positive model theory and compact abstract theories, Journal of Mathematical Logic 3 (2003), no. 1, 85–118.

   This paper introduces CATS - compact abstract theories - by which Ben Yaacov axiomatizes the type functors properties from first order logic (type spaces are compact, projections onto fewer variables are continuous maps, etc.) and develops a satisfactory abstract model theory. As one of the early motivations of this work, hyperimaginary sorts are seen to be capturable as CATS.

7. Ben Yaacov, I., Uncountable dense categoricity in cats, JSL 70 (2005), no. 3, 829–860.

   This is the transition paper. Although the intention of the paper was to prove the analogue of Morley's theorem in the context of CATS (which it does), the most important result that it contains is that every Hausdorff CAT is metric. It is a very short step from here to more or less abandoning the formalism of CATS and moving to the now familiar continuous setting.

8. Ben-Yaacov,, I. Berenstein, A., Henson, C.W. and Usvyatsov, A., Model theory for metric structures, Expanded lecture notes for a workshop given in March/April 2005, Isaac Newton Institute, University of Cambridge

9. Ben Yaacov, I. and Usvyatsov, A., Continuous first order logic and local stability, Transactions of the AMS, 362 (2010), no. 10, 5213–5259.

   Although this paper was published after the Newton Institute paper on continuous model theory, it was in circulation first and it is where I learned continuous logic. As the first paper in continuous model theory proper, it is interesting to read the introduction and the first section where the author's explain the choice of quantifiers in an abstract manner. The paper also contains the now almost totally forgotten forced limit construction.

10. Ben Yaacov, I. and Pedersen, A.P., A proof of completeness for continuous first-order logic, JSL 75 (2010), no. 1, 168–190.

    Here is an explicitly worked out proof system for continuous logic. We use it in our sketch of how CEP would allow us to run formal proofs from the recursive axioms of II_1 factors in order to obtain upper bounds for universal sentences.

11. Hart, B., An introduction to continuous model theory, appearing in the volume "Model theory of operator algebras" as part of DeGruyter's Logic and its Application Series, 2023.

    This is a concise introduction to continuous model theory which emphasizes its application to operator algebras and at the same time highlights the relationship between ultraproducts and definable sets in the continuous setting. It was meant to be the launching point of a book on continuous model theory which hopefully will be produced by the end of my sabbatical.

2 Some references for functional analysis and operator algebra

1. Pedersen, G. K., Analysis now. Springer-Verlag, 1989.

   If you are new to the study of functional analysis and/or operator algebras, this is an excellent introduction. It might be hard to come by but you should be able to find a pdf or two floating around.

2. J. von Neumann, Approximative properties of matrices of high finite order, Portugaliae mathematica, 3 (1942), 1–62.

   This paper contains extremely interesting calculations involving matrices, the relationship between the 2-norm and the operator norm and limiting properties as the size of the matrices tend to infinity. Ultraproducts did not appear until 10 or more years later but these results can be interpreted as the study of ultraproduct of matrix algebras.

3. Goldbring, I. A gentle introduction to von Neumann algebras for model theorists, notes from a graduate course, 2013. www.math.uci.edu/ isaac/vNanotes.pdf

   These notes, from Goldbring's continuous model theory course, helped a generation of graduate students learn about von Neumann algebras and their associated model theory.

4. Szabó, G., Introduction to C*-algebras. In I. Goldbring (Ed.), Model theory of operator algebras, 1–32, 2023, De Gruyter.

5. Ioana, A., An introduction to von Neumann algebras. In I. Goldbring (Ed.), Model theory of operator algebras 43–82, 2023, De Gruyter.

6. Anantharaman, C. and Popa, S., An introduction to II_1 factors, book available on Popa's website, www.math.ucla.edu/ popa/Books/IIun.pdf

   This book is one of the definitive sources for material related to $\mathrm{II}_!$ factors. Although it has never been published it remains available on Popa's website.

3 Model theory and operator algebra

1. Farah, I., Hart, B., and Sherman, D., Model theory of operator algebras I: Stability. Bull. of the LMS, 45(4), 825–838, 2013.

   This paper started the interaction between model theory and operator algebra. It was written with an operator algebra focus and contains a refutation of the McDuff conjecture i.e. it shows that it is consistent that for any separable II_1 factor has non-isomorphic ultrapowers with respect to ultrafilters on N.

2. Farah, I., Hart, B., and Sherman, D.. Model theory of operator algebras II: Model theory. IJM, 201, 477–505, 2014.

   This is a companion paper to the previous one which highlights the necessary model theory necessary to show that all theories of II_1 factors are unstable. It contains an explicit axiomatization of the classes of tracial von Neumann algebras and C*-algebras in continuous logic.

3. Farah, I., Hart, B., and Sherman, D., Model theory of operator algebras III: Elementary equivalence and II_1 factors. Bull. of the LMS, 46(3), 609–628, 2014.

   The focus of this paper is to enumerate properties of operator algebras that flow from the first two papers and highlight potential future work. It contains a discussion of how to see that matricial ultraproducts do not have property Gamma follows from von Neumann's 1942 Portuguese journal paper.

4. Farah, I., Goldbring, I., Hart, B. and Sherman, D., Existentially closed II_1 factors, Fund. Math. 233, 173–196, 2016.

5. Farah, I., Hart, B., Lupini, M., Robert, L., Tikuisis, A., Vignati, A., and Winter, W., Model theory of C*-algebras, Memoirs of the AMS, (1324). American Mathematical Society, 2021.

   This book grew out of a 2014 meeting in Muenster. It contains very detailed accounts of how to capture nuclear algebras in infinitary continuous model theory and regularity properties associated with the Elliott programme. It also contains more foundational material about subjects such as model theoretic forcing, imaginaries in continuous model theory and conceptual completeness.

6. Goldbring, I., Model Theory of Operator Algebras. De Gruyter, 2023.

   This book contains a host of interesting articles on the state of the subject at the time. All the articles are freely available on the arxiv.

4 Some instances of the use of ultraproducts outside of model theory

1. D. Sherman, Notes on automorphisms of ultrapowers of II_1 factors, Studia Mathematica 195 (2009), 201–217.

2. F. B. Wright, A reduction for algebras of finite type, Ann. of Math. (2) 60 (1954), 560–570.

   It can be argued that this paper is the first paper in which the ultraproduct construction is used in the study of von Neumann algebras. Out of the work described here, one can obtain the ultraproduct for tracial von Neumann algebras.

3. S. Sakai, The Theory of W*-algebras, lecture notes, Yale University, 1962.

4. D. McDuff, Central sequences and the hyperfinite factor, Proc. London Math. Soc. (3) 21 (1970), 443–461.

5. D. Dacunha-Castelle, J.Ł. Krivine, Application des ultraproduits à l'étude des espaces et algèbres de Banach, Studia Math. 41(1972), 315–334.

6. M. Gromov, Groups of polynomial growth and expanding maps, Publ. Math. IHES, 53 (1981), 53–78.

   Gromov introduces the notice of asymptotic cones in order to prove his theorem that every group with polynomial growth is nilpotent by finite. Asymptotic cones can be seen as an example of an ultraproduct contstruction (see van den Dries-Wilkie)

7. L. van den Dries and A. Wilkie, Gromov's theorem on groups of polynomial growth and elementary logic, Journal of Algebra, vol. 89 (1984), 349–374.

   Van den Dries and Wilkie couch Gromov's construction of the asymptotic cone in terms of ultraproducts of metric spaces which provides an alternative proof of Gromov's theorem.

8. A. Connes, Classification of injective factors: cases II_1, II_∞, III_λ, $\lambda \neq 1$, Ann. of Math. (2) 104 (1976), 73–115.

9. U. Groh, Uniform ergodic theorems for identity preserving Schwarz maps on W*-algebras, J. Operator Theory. vol. 11 no. 2, 1984, 395–404.

   This paper includes the genesis of ultraproduct construction of for arbitrary von Neumanns based on the idea that since von Neumann algebras have Banach space preduals, one can obtain an ultraproduct by taking the ultraproducts of the preduals and then considering the dual. This generates a huge algebra but it can be captured as a model theoretic ultraproduct. It is often called the Groh-Raynaud ultraproduct because of the later work of Raynaud which pointed out in an explicit way how Groh's work leads to ultraproducts.

10. A. Ocneanu, Actions of discrete amenable groups on von Neumann algebras, Lecture Notes in Mathematics (1138) Springer, 1985.

   The ultraproduct for W*-probability spaces, a generalization of the tracial case, first appears in this paper by Ocneanu.

5 MIP* and computability in continuous model theory

1. Z. Ji, A. Natarajan, T. Vidick, J. Wright, and H. Yuen, MIP*=RE, arxiv preprint, arxiv: 2001.04383, 2020.

   This paper contains the original proof that MIP*=RE, a landmark result in complexity theory showing the power of quantum entanglement. The proof has been modified and simplified since this original paper but this remains the only source for the full proof. It also sketches how MIP*=RE settles the Tsirelson problem as well as Kirchberg's QWEP conjecture and the Connes embedding problem.

2. B. S. Tsirelson. Some results and problems on quantum bell-type inequalities. Hadronic Journal Supplement, 8(4):329–345, 1993.

3. M. Junge, M. Navascues, C. Palazuelos, D. Perez-Garcia, V.B. Scholz, and R.F. Werner. Connes' embedding problem and Tsirelson's problem. J. Math. Phys., 52(1), 2011.

4. N. Ozawa. About the Connes embedding conjecture. Japanese Journal of Mathematics, 8(1):147–183, 2013.

5. T. Fritz.,Tsirelson's problem and Kirchberg's conjecture. Reviews in Math. Physics, 24(05), 2012.

6. E. Kirchberg, On non-semisplit extensions, tensor products and exactness of group C*-algebras, Invent. Math. 112 (1993), 449–489.

   This is where the QWEP conjecture is originally stated. I didn't talk about this during my talks but it is a crucial bridge between CEP and the Tsirelson problem.

7. Goldbring, I. and Hart, B., The universal theory of the hyperfinite II_1 factor is not computable, BSL, 30 (2) 181–198, 2024.

This paper proves that the universal theory of $\mathcal{R}$ is undecidable using the coding of values of non-local quantum games into universal sentences a la MIP*=RE. It also contains associated undecidability results for interesting operator algebras such as $\mathcal{Q}$ and $\mathcal{Z}$. As well, it gives a outline of how embedding problems like CEP often give quite a bit stronger results. It concludes by given elementary proofs that the fact that the QWEP conjecture and Tsirelson problem both fail follows from the non-computability of the universal theory of $\mathcal{R}$.