# Lecture 3: Partial resolution in characteristic 0

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# Resolution exists

**Theorem (ℵ-Tëmkin-Włodarczyk)**

*In characteristic 0, a smooth functorial singularity invariant and, for every $X \subset Y$, a center $\overline{J}$ satisfying (a) $V(\overline{J})$ is the maximal locus of $\operatorname{inv}_X$ and (b) $\max \operatorname{inv}_X = \max \operatorname{inv}_{BX}$ exist.*

In particular, by the criterion of Lecture 1

**Theorem (Hironaka 1964)**

*Resolution in characteristic 0 exists.*

# Normal-crossings-preserving resolution

## Definition

- A point $p \in X$ is a normal-crossings point if on a small analytic neighborhood $X = V(x_1 \cdots x_k)$, where $x_i$ are parameters.
- The nc($k$)-locus is the locus where $k$-branches meet transversely, in this case $V(x_1, \ldots, x_k)$.
- The locus of normal-crossings points is denoted $X^{\mathrm{nc}}$.
- A normal-crossings-preserving resolution $f : X' \to X$ is a birational map where $f$ is an isomorphicm over $X^{\mathrm{nc}}$ and $X'^{\mathrm{nc}} = X'$.

# A normal-crossings-preserving resolution does not exist

- Consider $X = V(x^2 - y^2 z)$.
- We have $X^{\text{nc}} = X \setminus \{0\}$. Here the non-zero $z$-axis is the nc(2)-locus.
- We note that if $X' \to X$ is a normal-crossings-preserving resolution, it must preserve the $z$ axis.
- On the other hand, the normalization of $X'$ would give an étale cover of $C$, which cannot split the branches of $X$.

# A normal-crossings-preserving resolution exists

**Theorem (2026: Belotto da Silva–Bierstone, Włodarczyk, see also Abramovich–Temkin, B-Bernard-B)**

*Let $X$ be a pure dimensional variety over a field of characteristic 0. There is a functorial stack theoretic Normal-crossings-preserving resolution $X' \to X$.*

Note that it says “stack-theoretic”.

# The umbrella again

- Consider $X = V(x^2 - y^2 z)$.
- We defined the center $(x^2, y^3, z^3)$ with reduced center $\overline{J} = (x^{1/3}, y^{1/2}, z^{1/2})$.
- The change of variables is $x = s^3 x', y = s^2 y', z = s^2 z'$.
- The resulting equation is $s^6(x'^2 - y'^2 z')$ with proper transform $X' = V(x'^2 - y'^2 z')$.
- Since we remove $V(x', y', z')$ we have that $X'$ is normal-crossings!

# An nc(2)-preserving resolution exists (sketch)

- Let's upgrade the example to an nc(2)-preserving resolution (in characteristic 0).
- Note that the invariant of an nc(2) hypersurface singularity is $(2,2)$.
- We make the simplifying assumption that $X \subset Y$ is a hypersurface in a smooth variety.
- We may use Theorem 1 and blow up as long as the maximal invariant is $\leq (2,2)$.
- But a singularity of invariant $(2,2)$ is nc(2), and any lower hypersurface invariant is of the form $(2,2,a_3,\ldots,a_k)$, which is regular in codimension 1.

**Exercise**

*prove this statement!*

- Therefore the nc(2) singularities can be left alone while the others are being resolved.

# Proof of Theorem 4: reduction to the inv-closed property

- We make the simplifying assumption that $X \subset Y$ is a hypersurface singularity $X = V(f)$.
- We know that $X^{\mathrm{nc}}$ is open.
- Say the highest singularity of $X^{\mathrm{nc}}$ is nc$(k)$, having invariant $K = (k, \ldots, k)$, of length $k$.
- As before we may assume the maximal invariant is $K$. Let $W_K \subset X$ be the locus of that invariant
- To mimic the nc(2) argument, we want to show that $X^{\mathrm{nc}} \cap W_K \subset W_K$ is open and closed.
- Since it is open, it suffices to show

**Proposition**

*$X^{\mathrm{nc}} \cap W_K \subset W_K$ is closed.*

# The inv-closed property

**Proposition**

$X^{\mathrm{nc}} \cap W_K \subset W_K$ *is closed.*

- We might as well assume $W_K$ is irreducible and generically normal-crossings.
- We have that $\mathcal{I}_{W_K} = \mathcal{D}^{k-1}(f)$, with local generators $t_1, \dots t_k$.
- We pick $w \in W_K$ and need to show it is normal-crossings.

# Cones and leading terms

- Let $f_k \in \oplus_{n\geq 0}\mathcal{I}_W^n/\mathcal{I}_W^{n+1}$ be the image of $f$
- in the graded coordinate ring $\oplus_{n\geq 0}\mathcal{I}_W^n/\mathcal{I}_W^{n+1}$ of the normal bundle $N_W \to W$ of $W$ in $Y$.
- $f_k$ describes the normal cone $C$ of $W$ in $X$.
- We denote its fiber over $w$ by $C_w$.
- It is defined in the vector space $N_w$ by the homogeneous form

$$(f_k \mod m_{w,W}).$$

# The invariant of $C_w$

- Separating terms depending on $t_i$ only, $f = f_k^0 + g$,
- where $g \in \mathcal{I}_W^k$ and $f_k^0 \in F[t_i]$ is homogeneous of degree $k$ and independent of other variables $y_i$.
- Computing derivatives we see that $\mathcal{D}_Y^{\leq(k-1)}(f_k^0)$ contains $k$ elements $t_i$ linearly independent modulo $\mathcal{I}_W^2$,
- and hence $\mathcal{D}_Y^{\leq(k-1)}(f_k^0)$ and $C_w = V(f_k^0)$ has invariant $K$ along $W$.

# $C_w$ has normal crossings

- Set $E = k(\eta)$. Since $X$ is normal-crossings at $\eta$, so is $C = V(f_k)$.
- Thus there exists a finite extension $E'/E$ such that $f_k = \prod_{i=1}^k \ell_i$ in $E'[\![\underline{t}]\!]$
- Setting $y_i = 0$, we get a decomposition of $f_k^0$ into linear factors $\ell_i^0$.
- However, $f_k^0$ has coefficients in $F$ (rather than $F[y_i]$),
- and hence the decomposition of $f_k^0$ has coefficients in $F' = E' \cap \overline{F}$
- which is a finite extension of $F$.
- Replacing $F$ by a finite extension we can assume that $f_k^0$ is a product of $k$ linear forms.
- Since the invariant of $C_w$ is $K$, these are linearly independent linear terms.

# $C$ has normal crossings: Hilbert scheme argument

- Consider the dual projective space $\mathbb{P}N_W^\vee$
- parametrizing hyperplanes in the fibers of $\mathbb{P}N_W \to W$. It is proper over $W$.
- The condition on a hyperplane being contained in the projective cone $\mathbb{P}C$ of $W$ in $X$ is closed,
- so defines a closed subscheme $H_X \subset \mathbb{P}N_W^\vee$.
- Hence the space $H_X$ of hyperplanes contained in $\mathbb{P}C$ is proper over $W$.
- Its generic fiber is finite and reduced of degree $k$, since $C_\eta$ is normal crossings by assumption.
- The argument above shows that the fiber of $H_X$ over $w$ has precisely $k$ reduced points.
- It follows that $H_X$ is finite étale over $w \in W$ of degree $k$.

# $C$ has normal crossings

- Replacing $Y$ by an étale neighborhood of $w$ we may ensure that $H_X$ becomes a disjoint union of $k$ copies of $W$.
- The universal homogeneous linear form on $N$ defining the corresponding family of hyperplanes defines elements $t_i$
- so that, on the chosen étale neighborhood, we have $f_k = t_1 \cdots t_k$,
- So $C = V(f_k)$ it has normal crossings, as needed.

# The blowup of $W_K$ is normal crossings: setup

*We apply induction on $k$, so that we may assume that the proposition holds for $nc(k')$ points, for all $X$ and all $k' < k$.*

- We show that the blowup of $X' \to X$ along $W$ is normal crossings transversal to the exceptional:
- Using our formalism the blowup has equation of the form $f' = t'_1 \cdots t'_k + sh(s, t', y)$,
- where $s$ is the exceptional variable and $t'_i$ are the transformed variables satisfying $t_i = st'_i$ for $i = 1 \ldots, k$.
- Consider a point $p_w$ on the exceptional $E_W = \{s = 0\}$ above $w$.
- Here $t'_1 = \cdots = t'_{k'} = 0$ but $t'_{k'+1}, \ldots, t'_k$ are units.

# The blowup of $W_K$ is normal crossings: the invariant

- On the one hand, such a point is a specialization of the point $p_\eta$ with the same coordinates $t'_i$ over $\eta$,
- which is an $\mathrm{nc}(k')$ point with invariant $K' = (k', \ldots, k')$.
- By upper semicontinuity $\mathrm{inv}_{X'}(p_w) \geq K'$.
- On the other hand, we claim that the invariant at $p_w$ of $X' = V(f')$ is $\leq K'$.
- Indeed, this follows from upper semicontinuity of inv in families, considering $s$ as a parameter.
- It follows that $\mathrm{inv}_{X'}(p_w) = K'$.

# The blowup of $W_K$ is normal crossings

- Thus $\mathrm{inv}_{X'}(p_w) = K'$.
- Being a specialization of an $\mathrm{nc}(k')$-point, the induction assumption implies it is an $\mathrm{nc}(k')$-point.
- Thus $X'$ is normal crossings along $E_W$.
- Moreover, the normal crossings branches of $X'$ meet $E_W$ transversally along $t'_i = 0$.
- Indeed, in the equation above, taking the derivative $\partial_{t'_2} \cdots \partial_{t'_{k'}}(f)$ and clearing units defines at $p_w$ a maximal contact element of the form $t'_1 + sh_1(s, t'_i, y_j)$.
- Permuting the variables one obtains $k'$ maximal contact elements of the form $t'_j + sh_j(s, t'_i, y_j)$ at $p_w$.
- This forces the normal crossings factors at $p_w$ to be of the same form.

# $X$ is normal crossings

- Passing to an open neighborhood of $E_W$ we may assume $X'$ is normal crossings,
- hence its normalization $\tilde{X}' \to X'$ is regular.
- The normalization $\tilde{X}'$ contains the $k$ *disjoint* loci $E^i_W := E_W \cap v(t'_i)$,
- whose union is the preimage of $W$.
- Since these loci are disjoint, an étale neighborhood of each one maps to a distinct branch of $X$ along $W$.
- It follows that $X$ has $k$ branches at $w$,
- and since the invariant is $K$, $X$ is normal crossings at $w$.