Lecture 2:
Resolution in characteristic 0 — how does it work?

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Resolution exists

Theorem (ℵ-Tëmkin-Włodarczyk)

In characteristic 0, a smooth functorial singularity invariant and, for every $X \subset Y$, a center $\bar{J}$ satisfying (a) $V(\bar{J})$ is the maximal locus of inv_X and (b) $\max \operatorname{inv}_X = \max \operatorname{inv}_{BX}$ exist.

In particular, by the criterion of Lecture 1 resolution exists.

Derivatives, order

Work with ideals instead of subvarieties — use $\mathcal{I}_X$.

- $\mathcal{D}^a(\mathcal{I}) := \{\nabla f \ : \ f \in \mathcal{I}, \nabla \in \mathcal{D}_X^{\leq a}\}$.
- Note: In characteristic 0 we have $\mathcal{D}^a(\mathcal{I}) = \mathcal{D}^1\mathcal{D}^{(a-1)}(\mathcal{I})$.
- $\operatorname{ord}_{\mathcal{I}}(p) \ := \min\{\, a \ : \ (\mathcal{D}^a(\mathcal{I}))_p = \mathcal{O}_{Y,p}\,\}$

Exercise

Show that $\operatorname{ord}_{\mathcal{I}}(p) \geq a \quad \Leftrightarrow \quad p \in V(\mathcal{D}^{a-1}(\mathcal{I}))$.
Conclude that $\operatorname{ord}_{\mathcal{I}}$ *is upper semicontinuous.*

Exercise

Show that $\mathcal{D}(\mathcal{I}\mathcal{O}_{Y\times\mathbb{A}^1}) = \mathcal{D}(\mathcal{I})\mathcal{O}_{Y\times\mathbb{A}^1}$.
Argue that $\operatorname{ord}_{\mathcal{I}}$ *is functorial for smooth morphisms (submersions).*

Exercise

Compute the derivatives and order of $(x^2 - y^2z)$ *and of* $(x^5 + x^3y^3 + y^{100})$ *at the origin.*

Maximal contact

- $x \in \mathcal{O}_{X,p}$ maximal contact if $x \in \mathcal{D}^{a-1}(\mathcal{I})_p$ and $\mathcal{D}^1(x) = \mathcal{O}_p$.

Exercise

Show that $V(x)$ smooth, and $V(x)$ contains all nearby points of order a, schematically.

- $V(x)$ is an excellent candidate for dimension induction.

Exercise

in characteristic 0 maximal contact exists.

Exercise

Find maximal contact elements for $(x^2 - y^2 z)$ and $(x^5 + x^3 y^3 + y^{100})$ at the origin.

Coefficient ideals

- Have seen: maximal contact $V(x)$ is a candidate for induction.
- What is the inductive version of $\mathcal{I}$?
- $\mathcal{I}, \mathcal{D}^1(\mathcal{I}), \ldots, \mathcal{D}^{a-1}(\mathcal{I})$ have orders $a, a-1, \ldots, 1$.
- Coefficient ideal $C(\mathcal{I}, a) := \sum_{\sum ib_i \geq a!} \prod_{i=1}^{a} (\mathcal{D}^{a-i}(\mathcal{I}))^{b_i}$.
- Coefficient $\mathbb{Q}$-ideal $C(\mathcal{I}, a)^{1/(a-1)!} = \sum_{i=1}^{a} \mathcal{D}^{a-i}(\mathcal{I})^{a/i}$.
- Notation: $\mathcal{I}[x] := (C(\mathcal{I}, a)|_{V(x)})^{1/(a-1)!}$.

Exercise

Compute the coefficient ideal and $\mathcal{I}[x]$ for $(x^2 - y^2 z)$, and compute $\mathcal{I}[x]$ for $(x^5 + x^3 y^3 + y^{100})$, both at origin.

Invariant and center

- Let $\mathrm{ord}_{\mathcal{I}}(p) = a_1$ with maximal contact x_1.
- Suppose $\mathrm{inv}_{\mathcal{I}[x_1]} = (a_2, \ldots, a_k)$ with center $J_{\mathcal{I}[x_1]} = (\bar{x}_2^{a_2}, \ldots, \bar{x}_k^{k})$
- (rescaling the invariant and center of $C(\mathcal{I}, a)|_{V(x)}$ by $1/(a-1)!$.)
- Let $x_2, \ldots, x_k$ be elements restricting to $\bar{x}_i$ on $V(x_1)$.
- Define $\mathrm{inv}_{\mathcal{I}}(p) := (a_1, a_2, \ldots, a_k)$.
- Define $J_{\mathcal{I}} := (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k})$.

Order invariants lexicographically, truncation is larger: $(2) > (2, 3)$.

Exercise

The set of values is well-ordered.

Exercise

Compute the invariant and center for $(x^2 - y^2 z)$ and for $(x^5 + x^3 y^3 + y^{100})$ at the origin.

Upper semicontinuity

- $a_1 = \mathrm{ord}_{\mathcal{I}}$ is upper semicontinuous,
- its maximum locus is contained in $V(x_1)$.
- By induction $\mathrm{inv}_{\mathcal{I}[x_1]}$ is upper semicontinuous on $V(x_1)$,
- so $\mathrm{inv}_{\mathcal{I}}$ upper semicontinuous.

Functoruiality of the invariant

- Have seen: $\mathrm{ord}_{\mathcal{I}}$ is functorial for smooth morphisms.
- Let us now choose x_1 — it remains maximal contact on smooth pullback.

Exercise

Prove this claim.

- By induction $\mathrm{inv}_{\mathcal{I}[x_1]}$ is functorial for smooth morphisms,
- hence $\mathrm{inv}_{\mathcal{I}}$ is functorial for smooth morphisms.

Note: not showing idependence of choices.

The maximal locus of the invariant

- The maximal locus of $\mathrm{ord}_{\mathcal{I}}$ lies inside $V(x_1)$.
- Inductively the maximal locus of $\mathrm{inv}_{\mathcal{I}[x_1]}$ is $V(x_2, \ldots, x_k)$.
- Hence the maximal locus of $\mathrm{inv}_{\mathcal{I}}$ is $V(x_1, x_2, \ldots, x_k)$, which is smooth.

We conclude that $\mathrm{inv}_{\mathcal{I}}$ is a smooth functorial singularity invariant satisfying condition (a) of the theorem.
It remains to show (b).

Admissibility

J is admissible for an ideal $\mathcal{I}$ if $v_J(f) \geq 1$ for all $f \in \mathcal{I}$.

Lemma

The center $J_{\mathcal{I}}$ is admissible for $\mathcal{I}$. If $f \in \mathcal{D}^i(\mathcal{I})$ then $v_J(f) \geq 1 - i/a_1$, and if $f \in C(\mathcal{I}, a_1)$ then $v_J(f) \geq (a_1 - 1)!$

Proof. Expand an element $f \in \mathcal{I}$ as

$$f = c_0 + c_1 x_1 + \cdots + c_{a_1 - 1} x_1^{a_1 - 1} + c_{a_1} x_1^{a_1} + \cdots,$$

where

$$c_i = \frac{1}{i!} \left(\frac{\partial^i f}{\partial x_1^i} \right)_{x_1 = 0} \in \mathcal{D}^i(\mathcal{I})|_{V(x_1)},$$

and use induction again.

The following is the real reason the center is well-defined:

Proposition

J is the unique center of maximal invariant which is admissible for $\mathcal{I}$.

Reduced center

There is a positive integer ℓ such that

$$\overline{J}_{\mathcal{I}} := J_{\mathcal{I}}^{1/\ell} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k}),$$

with w_i positive integers.
Note $a_i w_i = \ell$.
We use this rescaled center $\overline{J}_{\mathcal{I}}$ to prove the theorem.
We note that the associated monomial valuations satisfy $v_{\overline{J}} = \ell \cdot v_J$.

Transforms

- Write $\sigma : B \to Y$.
- If $f(x_1, \ldots, x_n) \in \mathcal{I}$ then

$$\sigma^* f = f(s^{w_1} x_1', \ldots, s^{w_k} x_k', x_{k+1} \ldots, x_k).$$

- admissibility gives $\sigma^* f = s^\ell f'$ for some regular function f'.
- If instead $f \in \mathcal{D}^i(\mathcal{I})$ then $\sigma^* f = s^{\ell - i w_1} f'$ for some regular function f'.
- So if $f \in C(\mathcal{I}, a_1)$ then $\sigma^* f = s^{(a-1)! \cdot \ell} f'$.
- Write

$$\begin{aligned} \mathcal{I}' &:= \{s^{-\ell} f : \qquad f \in \mathcal{I}\}, \\ \mathcal{D}(\mathcal{I})' &:= \{s^{-\ell + w_1} f : \quad f \in \mathcal{D}(\mathcal{I})\}, \\ C(\mathcal{I}, a_1)' &:= \{s^{-\ell(a_1 - 1!)} f : f \in C(\mathcal{I}, a_1)\}. \end{aligned}$$

Transforms of derivatives and coefficient ideals

Lemma

$(\mathcal{D}(\mathcal{I}))' \subset \mathcal{D}(\mathcal{I}')$.

Proof.

It suffices to show that $(\mathcal{D}(\mathcal{I}))' \subset \mathcal{D}_{B/\mathbb{A}^1}(\mathcal{I}')$.
$\mathcal{I}' = (s^{-\ell} f(s^{w_j} x_j) \ : \ f \in \mathcal{I})$ implies
$\mathcal{D}_{B/\mathbb{A}^1}(\mathcal{I}') = \left(\frac{\partial}{\partial x_i'}(s^{-\ell} f(s^{w_j} x_j')) \mid f \in \mathcal{I}, i = 1, \ldots, n\right)$,
and applying the chain rule this gives $\mathcal{D}_{B/\mathbb{A}^1}(\mathcal{I}') = \left(s^{-\ell+w_i} \frac{\partial f}{\partial x_i}(s^{w_j} x_j'))\right)$.
On the other hand $\mathcal{D}(\mathcal{I})' = \left(s^{-\ell+w_1} \frac{\partial f}{\partial x_i}(s^{w_j} x_j')\right)$ and $w_1 \geq w_i$

$$\mathcal{D}(\mathcal{I})' = \left(s^{-\ell+w_1} \frac{\partial f}{\partial x_i}(s^{w_j} x_j')\right) \subset \left(s^{-\ell+w_i} \frac{\partial f}{\partial x_i}(s^{w_j} x_j'))\right) = \mathcal{D}_{B/\mathbb{A}^1}(\mathcal{I}'),$$

♠

Transforms and coefficient ideals

Proposition

$\mathcal{I}'^{(a_1-1)!} \subset C(\mathcal{I}, a_1)' \subset C(\mathcal{I}', a_1)$.

Proof.

The inclusion $\mathcal{I}^{(a_1-1)!} \subset C(\mathcal{I}, a_1)$ implies an inclusion of transforms $\mathcal{I}'^{(a_1-1)!} \subset C(\mathcal{I}, a_1)'$.
The lemma above implies the second inclusion. ♠

We note that the invariant and center of $\mathcal{I}'^{(a_1-1)!}$ equals that of $C(\mathcal{I}', a_1)$. Proposition 2 says that this coincides with the invariant of $C(\mathcal{I}, a_1)'$.

Order and maximal contact element on B

Lemma

$\max_B \mathrm{ord}_{\mathcal{I}'} = a_1$ and wherever this is attained, x_1' is a maximal contact element.

- We assume that $\partial_{x_1}^{a_1}(\mathcal{I}) = \mathcal{O}$ and $x_1 \in \partial_{x_1}^{a_1-1}(\mathcal{I})$. It can be arranged by a linear change of variables or using derivatives with weight w_1.
- So there is an element $f \in \mathcal{I}$ such that $x_1 = \partial_{x_1}^{a_1-1}(f)$ and so $\partial_{x_1}^{a_1}(f) = 1$.
- Plugging in: $\partial_{x_1'}^{a_1}(s^{-\ell} f(s^{w_i} x_i')) = 1$, since $a_1 w_1 = \ell$.
- This means that the order everywhere is at most a_1.
- Where it is, we indeed get $x_1 = s^{-\ell+w_1}\partial_{x_1'}^{a_1-1}(f(s^{w_i} x_i'))$ so $x_1' = \partial_{x_1'}^{a_1-1}(s^{-\ell} f(s^{w_i} x_i'))$. So it is a maximal contact element.

The invariant of $\mathcal{I}'$

Lemma

$\max_B \operatorname{inv}_{\mathcal{I}'} = (a_1, \ldots, a_k)$

This implies (b), hence resolution of singularities.

Proof.

- By lemma the maximal order is a_1, attained within $H' = \{x_1' = 0\}$.
- $H' = BH$ for $H = \{x_1 = 0\}$, with center $\overline{J}_H = (x_2^{1/w_2}, \ldots, x_k^{1/w_k})$.
- By induction $\max \operatorname{inv}_{\mathcal{I}'[x_1']} = (a_2, \ldots, a_k)$
- (with center $J_{H'} = (x_2'^{a_2}, \ldots, x_k'^{a_k})$),
- so the maximal invariant if $\mathcal{I}'$ is indeed $(a_1, \ldots, a_k)$
- (with center $(x_1'^{a_1}, \ldots, x_k'^{a_k})$).

♠