Lecture 1:
Resolution in characteristic 0 — why does it work?

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Embedded resolution

- Work over $\mathbb{C}$. The main result is Hironaka's theorem.
- Given: singular closed subvariety X inside Y smooth.
- An embedded resolution is $f : Y' \to Y$ where
  - f is proper birational, and Y' smooth;
  - f is an isomorphism away from $Sing(X)$;
  - The proper transform X' of X is smooth.

Theorem (Hironaka 1964)

Such exists.

- I allow Y' and X' to be stacks (namely "moduli spaces")

Blowups

- I treat blowups following Rees, Fulton-MacPherson, Reid, Cox, Włodarczyk, Quek-Rydh.

- Let us blow up $\overline{J} = (x_1 \ \ , \dots, x_k \ \ )$ on $Y = \operatorname{Spec} k[x_1, \dots, x_n]$.
- Define the extended Rees algebra $\widetilde{R} := \mathcal{O}_Y[s, x_1', \dots, x_k']/(x_1 - s\ x_1',\ \dots,\ x_k - s\ x_k')$
- It has transformed variables x_i' and exceptional variable s.
- Its spectrum $B := \operatorname{Spec}_Y \widetilde{R}$ is the degeneration to normal cone of $\overline{J}$,
- with vertex $V(x_1', \dots, x_k')$.
- $\mathbb{G}_m$ acts on B via $t \cdot (s, x_1', \dots, x_k') = (t^{-1}s,\ tx_1', \dots, tx_k')$.
- The action stabilizes V, and the action on $B_+ := B \smallsetminus V$ is free.
- We finally define $Bl_{\overline{J}}(Y) := B_+/\mathbb{G}_m$.

Wait — what??

How is this related to the classical description?

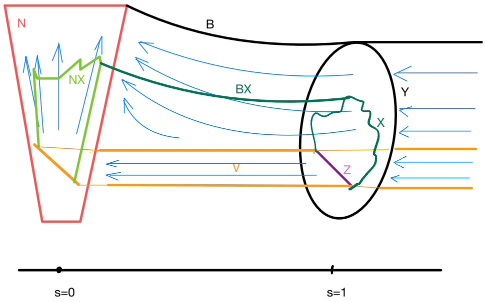
- $\widetilde{R} := \mathcal{O}_Y[s, x_1', \ldots, x_k']/(x_1 - s\ x_1',\ \ldots,\ x_k - s\ x_k')$
- $V(x_1', \ldots, x_k')$, $\qquad B_+ := B \setminus V$
- $\mathbb{G}_m$ acts on B via $t \cdot (s, x_1', \ldots, x_k') = (t^{-1}s,\ tx_1', \ldots, tx_k')$.
- $Bl_{\overline{J}}(Y) := \quad B_+/\mathbb{G}_m$.

- In the affine chart $\{x_i' \neq 0\} \subset B_+$ we have a $\mathbb{G}_m$-slice $x_i' = 1$:
- $x_i = s \quad \rightsquigarrow \quad x_j = x_i x_j'$, $\quad$ which describes the usual charts

Exercise

Verify this claim!

Picture



B is the degeneration to normal cone.
$s = 1$: just Y containing $Z := V(\bar{J})$
$s = 0 : \quad N_Z Y$
Action indicated with arrows
Also indicated what happens to a closed $X \subset Y$ containing $Z = V(\bar{J})$

Weighted blowups (Włodarczyk, Quek-Rydh.)

- $\overline{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ on $Y = \operatorname{Spec} k[x_1, \ldots, x_n]$.
- $\widetilde{R} := \mathcal{O}_Y[s, x_1', \ldots, x_k']/(x_1 - s^{w_1} x_1', \ \ldots, \ x_k - s^{w_k} x_k')$
- $B := \operatorname{Spec}_Y \widetilde{R}$ with vertex $V(x_1', \ldots, x_k')$.
- $\mathbb{G}_m$ acts on B via $t \cdot (s, x_1', \ldots, x_k') = (t^{-1}s, \ t^{w_1} x_1', \ldots, t^{w_k} x_k')$.
- Action stabilizes V, action on $B_+ := B \smallsetminus V$ has finite stabilizers.
- $Bl_{\overline{J}}(Y) := \left[B_+ \ / \ \mathbb{G}_m\right]$.

- the $\mathbb{G}_m$-slice $x_i' = 1$ is stabilized by $\boldsymbol{\mu}_{w_i}$:
- $x_i = s^{w_i} \quad \rightsquigarrow \quad x_j = s^{w_j} x_j'$, not the usual charts.

Weighted blowups — global meaning

- $\overline{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k}) \quad \longleftrightarrow \quad$ monomial valuation $v_{\overline{J}}(x_i) = w_i$.
- $\widetilde{R} = \bigoplus_{a \in \mathbb{Z}} R_a, \qquad R_a := \{j \in \mathcal{O}_Y \ : \ v_{\overline{J}}(f) \geq a\}$
- $V \ = \ V(\widetilde{R}_+), \qquad \widetilde{R}_+ := \left(\bigoplus_{a>0} R_a\right).$

Singularity invariants

- A singularity invariant is a rule assigning to a singular embedded variety a function, $X \subset Y \quad \rightsquigarrow \quad \mathrm{inv}_X : |X| \to \Gamma$ satisfying
  - Γ is a well-ordered set,
  - inv_X is upper-semicontinuous, and
  - $\mathrm{inv}_X(p) = \min \Gamma \quad \Leftrightarrow \quad p \in X$ is nonsingular.
- The invariant is functorial if whenever $\phi : Y_1 \to Y$ is smooth, with $X_1 = \phi^{-1}X$, we have

$$\mathrm{inv}_{X_1}(p_1) = \mathrm{inv}_X(\phi(p_1)).$$

- Smooth invariant if the maximal locus

$$\{p \in X \ : \ \mathrm{inv}_X(p) \text{ is maximal } \}$$

  in X is smooth.

Criterion

Assume we have a smooth, functorial singularity invariant inv.

Lemma

1. *Fix $X \subset Y$ and assume we have a center*

$$\overline{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$$

on Y which satisfies

(a) $V(\overline{J})$ *is the maximal locus of* inv_X *on X, and*

(b) $\max \mathrm{inv}_X = \max \mathrm{inv}_{BX}$.

Then

$$\max \mathrm{inv}_{Bl_{\overline{J}}(X)} < \max \mathrm{inv}_X.$$

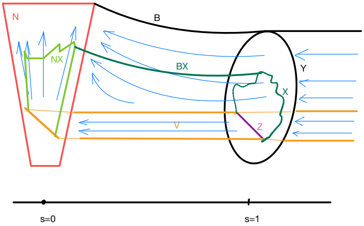
2. *Assume a center $\overline{J}$ satisfying these conditions exists for every resolution situation $X \subset Y$. Then resolution of singularities holds.*

Proof of criterion

Proof of 2: denoting $X' = Bl_{\bar{J}}(X)$, and then $X^{(n)} = Bl_{\bar{J}_{X^{(n-1)}}}(X^{(n-1)})$, the invariants drop, and must stop when $X^{(n)}$ is smooth.

Proof of 1:
Claim: The maximal locus W equals V.
With this, the invariant on B_+X drops. But $B_+X \to X'$ is smooth, so the invariant on X' drops.

Proof of claim



- $\{s \neq 0\} \to Y$ is smooth. So by functoriality and (a), $W \cap \{s \neq 0\} = V \cap \{s \neq 0\}$, with maximal invariant as that of X.
- By (b) the maximal invariant on B is the same, so by upper-semicontinuity and smoothness V is an connected component of W.
- By functoriality any point $b \in N_Z \cap W$ comes along with its orbit.
- The limit point of the orbit lies in V, contradicting that V is a connected component of W.

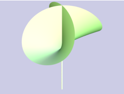
Resolution exists

Theorem (ℵ-Tëmkin-Włodarczyk)

In characteristic 0, a smooth functorial singularity invariant and, for every $X \subset Y$, a center $\bar{J}$ satisfying (a) *and* (b) *exist.*

- In particular, by 2 resolution exists.
- I am not aware of a proof of Hironaka's theorem going through such criterion without weighted blowups.
- Let us see some examples.

Weighted blowup of Whitney's umbrella $X = V(x^2 - y^2 z)$.



- The most singular point is the origin $p = V(x, y, z)$,
- x, y, z appear in degrees $2, 3, 3$ respectively.
- $\mathrm{inv}_p = (2, 3, 3)$, and $J = (x^2, y^3, z^3)$,
- *reduced* center $\overline{J} = (x^{1/3}, y^{1/2}, z^{1/2}) = J^{1/6}$ which we can blow up.

Exercise

Blow up and analyze the result.

Exercise

Draw the Newton polygon of $X = V(x^2 - y^2 z)$ *and of the center* (x^2, y^3, z^3). *What can you surmise on their relationship?*

Standard blowup of Whitney's umbrella

Consider the same X but with the blowup but with $\overline{J} = (x, y, z)$.

Exercise

Do this blowup!

Resolving $X = V(x^5 + x^3y^3 + y^{100})$

Exercise

Draw the Newton polygon of $X = V(x^5 + x^3y^3 + y^{100})$ *and of the center* (x^2, y^3). *What can you surmise on their relationship?*

Exercise

Blow up X *along* $\overline{J} = (x^{1/3}, y^{1/2})$