May 2018

J. Lamboley

Shape optimization under convexity constraint

Sheet 2 : Regularity

Exercice 1 (Ugly convex function)**.**

1. Build a subset of $[0,1]$ which is compact, with empty interior, and positive measure.
2. Define f such that $f'' = \mathbb{1}_K$. What can you say about f ?
3. We search for g such that $f + tg$ is convex for every t such that $|t|$ is small enough (we want to allow positive and negative values fot t).
   (a) Can you find g smooth ?
   (b) Can you find g not necessarily smooth ?
   (c) * Can we find g localized in a small set of $[0,1]$?

Exercice 2 (Minimization of λ_2 under volume and convexity constraint)**.** We are interested in

$$\min\left\{\lambda_2(\Omega),\ \ \Omega \text{ open convex } \subset \mathbb{R}^N,\ \ |\Omega| = V_0\right\}. \tag{1}$$

1. Show that if we drop the convexity constraint, any union of two disjoint balls of volume $V_0/2$ is a solution.
2. Show that there exists a solution to (1).
3. Show that (1) is (in a sense to specify) equivalent to the following problems :
$$\min\left\{\lambda_2(\Omega)|\Omega|^{2/N},\ \ \Omega \text{ convex bounded} \subset \mathbb{R}^N\right\},$$
   and
$$\min\left\{\lambda_2(\Omega) + \alpha|\Omega|,\ \ \Omega \text{ convex bounded} \subset \mathbb{R}^N\right\},$$
   for a suitable value of $\alpha > 0$.
4. Let Ω be open and convex in $\mathbb{R}^2$ ($N = 2$ to simplify), and assume $\partial\Omega$ has a corner at $x_0 \in \partial\Omega$. Denoting (u_1, u_2) the first two eigenfunctions of Ω, we know that $u_i(x) = o(|x|)$ in a neighborhood of $x_0, i = 1, 2$. We define Ω_ε the shape obtained by cutting the angle x_0 with height $\varepsilon > 0$ (in direction of the angle bisector at x_0).
   (a) Show that
$$\lambda_2(\Omega_\varepsilon) \leq \lambda_2(\Omega) + o(\varepsilon^2).$$
   Hint : use the min-max formulation of λ_2, and use (u_1, u_2) multiplied by a suitable cut-off function to get an upper bound for $\lambda_2(\Omega_\varepsilon)$)
   (b) Conclude that Ω cannot be a solution to (1).

Exercice 3 (Shape derivative)**.** If J is a shape functional, Ω a set, and $V : \mathbb{R}^N \to \mathbb{R}^N$ is smooth, we define

$$J'(\Omega).V = \lim_{t\to 0} \frac{J((Id + tV)(\Omega)) - J(\Omega)}{t}$$

1. Prove that for $|t|$ small and $V : \mathbb{R}^N \to \mathbb{R}^N$ smooth, then $Id + tV$ is bijective on $\mathbb{R}^N$.
2. Compute the shape derivative of the volume, and show that if Ω is smooth enough, then it can be written as an integral over $\partial\Omega$.