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Shape optimization under convexity constraint

Sheet 1 : Existence results and Newton's problem

Exercice 1 (Hausdorff metric). Let D be a fixed compact set in $\mathbb{R}^d$, and for K_1, K_2 two compact subsets of D, we define

$$d^H(K_1, K_2) := \|d_{K_1} - d_{K_2}\|_{\infty,D}, \qquad \left(\text{where } d_K(x) := \inf_{y\in K} \|x-y\|\right)$$

the Hausdorff 'distance' between K_1 and K_2 *(is it always well-defined ?)*.

1. Show that d^H is a metric on $\mathcal{K}(D) = \{K \text{ non-empty compact subset of } D\}$.
2. Show that $(\mathcal{K}(D), d^H)$ is compact *(Hint : apply Ascoli Theorem to $(d_{K_n})_{n\in\mathbb{N}}$, and denote f the limit of a subsequence. Then prove that $f = d_K$ where $K = \{f = 0\}$)*.

Exercice 2 (Newton's problem). Let D be the unit disk in $\mathbb{R}^2$ and $M > 0$. We consider

$$\min\left\{F(u) := \int_D \frac{1}{1+|\nabla u(x)|^2}dx, \quad u : \overline{D} \to [0, M], \quad u \text{ continuous and concave}\right\}. \tag{1}$$

1. Show that the problem would have no solution if we dropped either the concavity constraint or the upper bound constraint M.
2. Show that if u is solution of (1), then there are some points $x_0, x_1 \in \overline{D}$ such that $u(x_0) = 0$ and $u(x_1) = M$ *(in other words, the solution must saturate the bound constraints)*.
3. (a) Expand $F(u + tv)$ at second order in t close to 0.
   (b) Let u be a solution of (1). Let $x_0 \in D$, and assume there is a neighborhood ω of x_0 where u is C^2, uniformly convex, and do not touch the bound M.
      i. Show that $F''(u).(v, v) \geq 0$ for any v smooth and supported in ω.
      ii. * Deduce a contradiction using well-chosen test functions.
   (c) What can we deduce about the radial solution found by Newton ?

Exercice 3 (Hausdorff distance and convex sets). 1. Is the volume continuous for the Hausdorff distance, among compact **convex** sets ?

2. We define the perimeter of K as $\mathcal{H}^{N-1}(\partial K)$ ($\mathcal{H}^{N-1}$ is the $N-1$ dimensional Hausdorff measure in $\mathbb{R}^N$) **if** K has nonempty interior.
   (a) Show that the perimeter is monotone with respect to inclusion among convex sets (with nonempty interior) ; is it still valid if we drop the convexity ?
   (b) If K_n converges to K for the Hausdorff distance and if all sets are convex with nonempty interior, can we say that the perimeter of K_n converges to the one of K ?
   (c) Explain what happens when K_n converges to a set with empty interior (flat) and what is the natural extension of the perimeter for flat sets.