EXERCISES

The first three exercises prove an estimate on the number of eigenvalues contained in an interval that has an error effective in the geometry of the surface.

Exercise 1. For $t > 0$, let $h_t(r) = e^{-t(\frac{1}{4}+r^2)}$. Let $p_t(\varrho)$ be the inverse Selberg transform of $h_t(r)$. Show that p_t satisfies the following estimate: there exists a constant $C > 0$ such that

$$0 \leq p_t(\varrho) \leq \frac{C}{t} e^{-\frac{\varrho^2}{8t}}.$$

Hint: Split the integration domain over $[\varrho, 2\varrho]$ and $[2\varrho, \infty)$ and use the mean value theorem in the first domain.

Exercise 2. Let X be a closed and connected hyperbolic surface. Using the Selberg trace formula, show that there exists a constant $C > 0$ such that for any $t, T > 0$ with $\lfloor 2T \rfloor \geq \lfloor 16t \rfloor$,

$$\frac{1}{\mathrm{Vol}(X)} \sum_{j=0}^{\infty} e^{-t\lambda_j} = \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-t(\frac{1}{4}+r^2)} r \tanh(\pi r) \mathrm{d}r + O\left(\frac{1}{t \min\{1, \mathrm{InjRad}(X)^2\}} \left(e^{-2T} + e^{2T} \frac{\mathrm{Vol}(X_{\leq T})}{\mathrm{Vol}(X)}\right)\right).$$

Hint: Use the trace formula with the heat kernel from exercise 1:

$$\sum_{j=0}^{\infty} e^{-t(\frac{1}{4}+r_j^2)} = \frac{\mathrm{Vol}(X)}{4\pi} \int_{-\infty}^{\infty} e^{-t(\frac{1}{4}+r^2)} r \tanh(\pi r) \mathrm{d}r + \int_D \sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} p_t(d(z, \gamma z)) \mathrm{dVol}_{\mathbb{H}}(z).$$

Then split the integration domain for the term involving the kernel over points with injectivity radius at most T and larger than T. For points with injectivity radius larger than T, the sum over Γ can be rewritten as $\sum_{m=\lfloor 2T \rfloor}^{\infty} \sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}: m \leq d(z,\gamma z) \leq m+1}$. Then use bounds on the number of group elements with $d(z, \gamma z) \leq m + 1$ similar to in the lecture. For points with injectivity radius smaller than T, work similarly as before but consider two different regions: $\mathrm{InjRad}(X) \leq d(z, \gamma z) \leq \lfloor 2T \rfloor$ and $d(z, \gamma z) \geq \lfloor 2T \rfloor$.

Exercise 3. Suppose that f is a continuous function with compact support on $\mathbb{R}_{\geq 0}$. Then one can approximate f uniformly by finite linear combinations of functions of the form e^{-tx} for $t > 0$. Given an interval $I = [a, b] \subset (\frac{1}{4}, \infty)$, fix $\varepsilon = \frac{1}{8}|I|$ and define

$$f(x) = \begin{cases} 0 & \text{if } x \notin [a, b], \\ \frac{1}{\varepsilon}(x - a) & \text{if } x \in [a, a + \varepsilon], \\ 1 & \text{if } x \in [a + \varepsilon, b - \varepsilon], \\ \frac{1}{\varepsilon}(b - x) & \text{if } x \in [b - \varepsilon, b]. \end{cases}$$

By the previous statement, for any $\delta > 0$ we can find a function $g(x) = \sum_{k=1}^{n} a_k e^{-t_k x}$ such that

$$\|g(x) - f(x)e^x\|_\infty < \delta, \tag{0.1}$$

where the t_k and n depend only on the interval I and δ. For X a closed and connected hyperbolic surface of genus $g \geq 2$, we have

$$\frac{N(X, I)}{\mathrm{Vol}(X)} \geq \frac{1}{\mathrm{Vol}(X)} \sum_{j=0}^{\infty} f(\lambda_j). \tag{0.2}$$

(1) Let $T = \frac{1}{24}\log(g)$ so that for g sufficiently large, $\lfloor 2T \rfloor \geq \max\{1, \lfloor 16t_k \rfloor\}$ for each k. Then using exercise 2, show that

$$\frac{N(X,I)}{\mathrm{Vol}(X)} \geq \frac{1}{2\pi}\int_0^\infty e^{-(\frac{1}{4}+r^2)} r\tanh(\pi r) g(\tfrac{1}{4}+r^2)\mathrm{d}r - \frac{\delta}{2\pi}\int_0^\infty e^{-(\frac{1}{4}+r^2)} r\tanh(\pi r)\mathrm{d}r$$
$$+ O\left(\frac{1}{\min\{1, \mathrm{InjRad}(X)^2\}}\left(g^{-\frac{1}{12}} + \log(g) g^{\frac{1}{12}} \frac{\mathrm{Vol}(X_{\leq \frac{1}{24}\log(g)})}{\mathrm{Vol}(X)}\right)\right).$$

Now assume that X satisfies the following estimates when the genus is sufficiently large:

$$\frac{\mathrm{Vol}(X_{\leq \frac{1}{24}\log(g)})}{\mathrm{Vol}(X)} \leq g^{-\frac{5}{12}}, \text{ and } \mathrm{InjRad}(X) \geq g^{-\frac{1}{30}}.$$

(2) Use part (1) as well as the estimate (0.1) to deduce that there exist constants $c, C(I) > 0$ such that when the genus of X is sufficiently large,

$$\frac{N(X,I)}{\mathrm{Vol}(X)} \geq C(I) - O(g^{-c}). \tag{0.3}$$

Remark. We will see in the minicourse that the geometric estimates assumed for X occur with probability tending to 1 as $g \to \infty$.

Exercise 4. This exercise will describe the necessary steps to show that quantum ergodicity in the large eigenvalue aspect is equivalent to demonstrating convergence to zero of the spectral averages.

(1) Prove that the intersection of finitely many density one sets is also density one. (by induction it suffices to just consider two sets)

(2) Suppose that $(F_k)_{k\geq 1}$ is a sequence of subsets of $\mathbb{N}$ with $F_{k+1} \subseteq F_k$ for all $k \geq 1$. Furthermore, suppose that $(N_k)_{k\geq 1}$ is an increasing sequence of natural numbers such for which

$$\frac{1}{N}|\{j \in \{1,\ldots,N\} : j \in F_k\}| \geq 1 - \frac{1}{k}, \quad \text{whenever } N \geq N_k.$$

Prove that there exists a set F with density one such that $F \cap [N_k, \infty) \subseteq F_k$ for all $k \geq 2$. In particular, this holds whenever $(F_k)_{k\geq 1}$ is a non-increasing sequence of sets of density one.

(3) Prove that if $(b_k)_{k\geq 1}$ is a bounded sequence of complex numbers such that

$$\frac{1}{N}\sum_{k=1}^N |b_k| \to 0, \quad \text{as } N \to \infty,$$

then there exists a set F of density one such that $b_k \to 0$ as $k \to \infty$ $k \in F$. Hint: Use Chebyshev's inequality along with the previous exercise.

(4) Now suppose that $(a_k)_{k\geq 1} \subseteq C^\infty(X)$ is a dense collection in $C_c(X)$. Suppose further that for each $k \geq 1$, there is a density one set $F_{a_k} \subseteq \mathbb{N}$ for which

$$\left|\langle a_k\psi_j, \psi_j\rangle - \frac{1}{\mathrm{Vol}(X)}\int_X a_k(x)\mathrm{dVol}_X(x)\right| \to 0,$$

as $j \to \infty$ along $j \in F_k$. Using parts 1 and 2, show that there exists a density one set F such that for any $a \in C_c(X)$,

$$\left|\langle a\psi_j, \psi_j\rangle - \frac{1}{\mathrm{Vol}(X)}\int_X a(x)\mathrm{dVol}_X(x)\right| \to 0,$$

as $j \to \infty$ along $j \in F$.