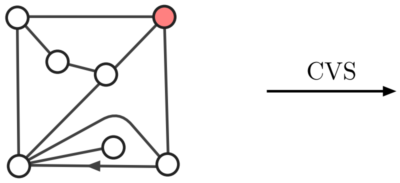
Geometry of random planar maps and genus-0 hyperbolic surfaces

ISM school: Geometry and spectra of random hyperbolic surfaces (University of Montréal)
Exercises by T. Budd for tutorial on 15 June 2023

Exercise 1.1: Tree bijection for quadrangulations

A (planar) quadrangulation is a rooted planar map with all faces of degree 4. For these maps the BDFG bijection specializes to the Cori-Vauquelin-Schaeffer (CVS) bijection: since all unlabeled black vertices in the mobile are of degree two, they can be removed by merging their adjacent edges and one ends up with a $\mathbb{Z}$-labeled (rooted) plane tree.

a) Determine the $\mathbb{Z}$-labeled plane tree associated to the following pointed quadrangulation.



b) What are the allowed label increments along the edges of the tree? Prove that the number of quadrangulations (not pointed) with n faces is

$$Q_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}. \tag{1.1}$$

c) Let $\mathfrak{q}_n$ be a uniform random quadrangulation with n faces and v_1, v_2 a uniform pair of distinct random vertices of $\mathfrak{q}_n$. Let d_1 be the graph distance between v_1 and the furthest end of the root edge and d_2 the graph distance between v_1 and v_2. Prove that

$$\mathbb{E}[d_2 - d_1] = 0, \qquad \lim_{n\to\infty} \mathbb{E}\left[\left(\frac{d_1 - d_2}{n^{1/4}}\right)^2\right] = \frac{\sqrt{\pi}}{3}. \tag{1.2}$$

Hint: Interpret $d_2 - d_1$ on the level of the tree. You may use the fact that the expected height (graph distance to the root vertex) of a uniform vertex in a uniform plane tree with n edges is asymptotic to $\frac{1}{2}\sqrt{\pi n}$.

This suggests that graph distances between typical vertices in $\mathfrak{q}_n$ are of order $n^{1/4}$.

Exercise 1.2: Generating functions of maps with boundaries

Recall that the partition function

$$F_0^{\mathrm{m}}(t, q) = \sum_{\text{rooted maps } \mathfrak{m}} \frac{1}{2|E(\mathfrak{m})|} t^{|V(\mathfrak{m})|} \prod_{f \in F(\mathfrak{m})} q_{\deg f/2}. \tag{2.1}$$

of bipartite planar maps with face weights $q = (q_1, q_2, \ldots)$ and vertex weight t is given by

$$F_0^{\mathrm{m}}(t, q) = \frac{1}{2} \int_0^R \frac{\mathrm{d}r}{r} \left[(g_q(r) - t)^2 - (r - t)^2 \mathbb{1}_{\{r<t\}}\right] \tag{2.2}$$

where

$$g_q(r) = r - \sum_{k=1}^{\infty} q_k \binom{2k-1}{k} r^k, \tag{2.3}$$

and $R(t,q) = \frac{t}{1-q_1} + O(t^2)$ is the power series solution to $g_q(R) = t$. Show that the pointed disk function

$$W_\bullet^{(\ell)}(t,q) = 2\ell \frac{\partial^2 F_0^{\mathrm{m}}}{\partial t\, \partial q_\ell} \tag{2.4}$$

is given by

$$W_\bullet^{(\ell)}(t,q) = \binom{2\ell}{\ell} R^\ell. \tag{2.5}$$

Hint: Use that $g_q(R) = t$ repeatedly!

Exercise 1.3: Size of Boltzmann planar maps

Let us choose some real number $q_1, q_2, \ldots \geq 0$ such that only finitely many of the q_k are nonzero (and at least one of $q_2, q_3, \ldots$ is positive). Then for t sufficiently small

$$R(t,q) = \sum_{\mathfrak{m} \in \vec{M}_0^\bullet} t^{|V(\mathfrak{m})|-1} \prod_{f \in F(\mathfrak{m})} q_{\deg f/2} < \infty. \tag{3.1}$$

Recall that R solves the equation $g_q(R) = t$, where

$$g_q(r) = r - \sum_{k=1}^{\infty} q_k \binom{2k-1}{k} r^k. \tag{3.2}$$

a) Prove that the coefficient of t^n in $R(t,q)$, denoted $[t^n]R(t,q)$, satisfies

$$[t^n]R(t,q) \overset{n\to\infty}{\sim} C t_*^{-n} n^{-3/2} \qquad \text{for some } t_*, C > 0. \tag{3.3}$$

Hint: You may use the following *Transfer theorem.* If $f(x)$ is a power series with positive coefficients that is analytic on $[0, x_*)$ and for $c > 0$, $\alpha \in (0,1)$,

$$f(x) = f(x^*) - c(x^* - x)^\alpha + o((x - x^*)^\alpha), \quad \text{then} \quad [x^n]f(x) \overset{n\to\infty}{\sim} \frac{c}{\Gamma(-\alpha)} x_*^{-n} n^{-\alpha-1}. \tag{3.4}$$

b) Recall our definition of the rooted pointed (t,q)-Boltzmann planar map as the probability distribution

$$\mathbb{P}(\mathfrak{m}) = \frac{1}{R(t,q)} t^{|V(\mathfrak{m})|-1} \prod_{f \in F(\mathfrak{m})} q_{\deg f/2} \tag{3.5}$$

on rooted, pointed maps $\mathfrak{m}$. Prove that the number of vertices in such a map obeys

$$\mathbb{P}(|V(\mathfrak{m})| = n+1) = \frac{C}{R} \left(\frac{t}{t_*}\right)^n n^{-3/2}. \tag{3.6}$$

In particular, $\mathbb{E}[|V(\mathfrak{m})|] < \infty$ if $t < t_*$ (*subcritical*) and $\mathbb{E}[|V(\mathfrak{m})|] = \infty$ if $t = t_*$ (*critical*).

Exercise 1.4: Generating function of ψ-class intersection numbers

In this exercise you will prove the fact stated in the lecture that the solution $F_0(t_0, t_1, \ldots) = t_0^3 + \cdots$ to the *string equation*

$$\frac{\partial F_0}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F_0}{\partial t_i}, \tag{4.1}$$

is given by

$$F_0(t_0, t_1, \ldots) = \frac{1}{2} \int_0^{u_0} Z(r)^2 \mathrm{d}r, \tag{4.2}$$

where $u_0(t_0, t_1, \ldots) = t_0 + \cdots$ is the formal power series solution to

$$Z(u_0) = 0, \qquad Z(r) := r - \sum_{k=0}^{\infty} t_k \frac{r^k}{k!}. \tag{4.3}$$

a) Define for $p \geq 1$ the power series

$$f_p(t_0, t_1, \ldots) = \sum_{k=0}^{\infty} t_{k+p} \frac{u_0^k}{k!}. \tag{4.4}$$

Show that

$$\frac{\partial u_0}{\partial t_0} = \frac{1}{1-f_1}, \qquad \frac{\partial f_p}{\partial t_0} = \frac{f_{p+1}}{1-f_1}. \tag{4.5}$$

Hint: Compute $\frac{\mathrm{d}}{\mathrm{d}t_0} Z(u_0)$.

b) Make use of the string equation and (4.5) to show that

$$\frac{\mathrm{d}}{\mathrm{d}s} F_0(t_0 - s, f_1(s, t_1, t_2, \ldots), f_2(s, t_1, t_2, \ldots), \ldots) = -\frac{1}{2}(t_0 - s)^2 \frac{\partial u_0}{\partial t_0}(s, t_1, t_2, \ldots). \tag{4.6}$$

c) Integrate (4.6) from $s = 0$ to $s = t_0$ to prove (4.2).
Hint: Argue that $u_0(0, t_1, \ldots) = 0$ and $f_p(0, t_1, \ldots) = t_p$, and perform a change of integration variables $s \to r = u_0(s, t_1, \ldots)$.