EXERCISES 2
TRACE OF THE HEAT KERNEL AND ISOSPECTRALITY

(1) The heat equation on a compact hyperbolic surface $M = \Gamma\backslash\mathbb{H}$ asks for a solution $u : \mathbb{R}_+ \times M \to \mathbb{R}$ such that

$$\begin{cases} \partial_t u = \Delta u \\ u(0,x) = f(x) \end{cases}$$

with initial condition $f \in C^\infty(M)$. A first step in constructing a solution is to assume that $-\Delta f = \lambda f$ and consider

$$u(t,z) = \int_{\mathbb{H}} h_t(z,w) f(w) d\mu(w)$$

for some choice of point-pair invariant $h_t(z,w)$. Show that $\widehat{h_t}(\lambda) = e^{-t\lambda}$.

With Selberg's inversion formulas and some more work, one can then construct the heat kernel

$$H_t(z,w) = \sum_{\gamma\in\Gamma} h_t(z,\gamma w)$$

for M and solve the heat equation.

(2) Show that the trace of the heat kernel, i.e.,

$$\int_M H_t(z,z)\, d\mu(z) = \sum_{\lambda\in\mathrm{Spec}(-\Delta)} e^{-t\lambda}$$

completely determines the spectrum of $-\Delta$. (Hint: Consider $e^{rt}\,\mathrm{Tr}(H_t)$ as $t \to \infty$.)

Can we hear the shape of a drum? Namely, if M_1 and M_2 are isospectral compact hyperbolic surfaces, are they necessarily isometric?

For the rest of this problem sheet we will consider the covering diagram

$$\begin{tikzcd}
& M \arrow[dl, "H_1"'] \arrow[dd, "G"] \arrow[dr, "H_2"] & \\
M_1 \arrow[dr] & & M_2 \arrow[dl] \\
& M_0 &
\end{tikzcd}$$

with $M = \Gamma\backslash\mathbb{H}$, $M_i = \Gamma_i\backslash\mathbb{H}$, $\Gamma \lhd \Gamma_i$ normal of finite index for $i = 0,1,2$. The corresponding covering groups are $G = \Gamma_0/\Gamma$, $H_1 = \Gamma_1/\Gamma$, $H_2 = \Gamma_2/\Gamma$. The surfaces M_1 and M_2 are isometric if and only if Γ_1 and Γ_2 are isometric in $\mathrm{Isom}(\mathbb{H})$. By the previous exercise, we say that M_1 and M_2 are isospectral if they have the same heat trace.

(3) Show that
$$H_t^{M_0}(z,w) = \sum_{\gamma\in\Gamma_0} h_t(z,\gamma w) = \sum_{g\in G} H_t^M(z,gw).$$
and
$$\mathrm{Tr}(H_t^{M_0}) = \frac{1}{|G|}\sum_{g\in G}\int_M H_t^M(z,gz)\,d\mu(z) = \sum_{[g]}\frac{|[g]|}{|G|}\int_M H_t^M(z,gz)d\mu(z),$$
where the sum is over conjugacy classes of g in G.

(4) (Sunada's criterium for isospectrality [1]) Let $i=1,2$. Show that if $h_1,h_2\in H_i$ are conjugate in G then
$$\int_M H_t^M(z,h_1z)d\mu(z) = \int_M H_t^M(z,h_2z)d\mu(z)$$
hence
$$\mathrm{Tr}(H_t^{M_i}) = \sum_{[g]}\frac{|[g]\cap H_i|}{|H_i|}\int_M H_t^M(z,gz)d\mu(z).$$
Conclude that M_1 and M_2 are isospectral if
$$|[g]\cap H_1| = |[g]\cap H_2|$$
for every $g\in G$. If this holds we say that H_1 and H_2 are *almost conjugate* in G. Check that if H_1 and H_2 are conjugate in G then they are almost conjugate.

(5) Show that $H_1 = \left(\begin{smallmatrix}1&*&*\\0&*&*\\0&*&*\end{smallmatrix}\right)$, $H_2 = H_1^T$ are almost conjugate but not conjugate in $G = \mathrm{SL}_3(\mathbb{Z}/2\mathbb{Z})$.

References

[1] T. Sunada, *Riemannian coverings and isospectral manifolds*. Ann. Math. 121 (1985).