

EXERCISES 2
TRACE OF THE HEAT KERNEL AND ISOSPECTRALITY

- (1) The heat equation on a compact hyperbolic surface $M = \Gamma \backslash \mathbb{H}$ asks for a solution $u : \mathbb{R}_+ \times M \rightarrow \mathbb{R}$ such that

$$\begin{cases} \partial_t u = \Delta u \\ u(0, x) = f(x) \end{cases}$$

with initial condition $f \in C^\infty(M)$. A first step in constructing a solution is to assume that $-\Delta f = \lambda f$ and consider

$$u(t, z) = \int_{\mathbb{H}} h_t(z, w) f(w) d\mu(w)$$

for some choice of point-pair invariant $h_t(z, w)$. Show that $\widehat{h}_t(\lambda) = e^{-t\lambda}$.

With Selberg's inversion formulas and some more work, one can then construct the heat kernel

$$H_t(z, w) = \sum_{\gamma \in \Gamma} h_t(z, \gamma w)$$

for M and solve the heat equation.

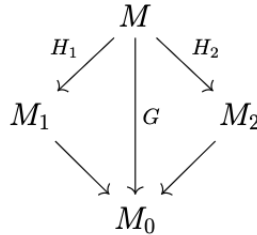
- (2) Show that the trace of the heat kernel, i.e.,

$$\int_M H_t(z, z) d\mu(z) = \sum_{\lambda \in \text{Spec}(-\Delta)} e^{-t\lambda}$$

completely determines the spectrum of $-\Delta$. (Hint: Consider $e^{rt} \text{Tr}(H_t)$ as $t \rightarrow \infty$.)

Can we hear the shape of a drum? Namely, if M_1 and M_2 are isospectral compact hyperbolic surfaces, are they necessarily isometric?

For the rest of this problem sheet we will consider the covering diagram



with $M = \Gamma \backslash \mathbb{H}$, $M_i = \Gamma_i \backslash \mathbb{H}$, $\Gamma \triangleleft \Gamma_i$ normal of finite index for $i = 0, 1, 2$. The corresponding covering groups are $G = \Gamma_0/\Gamma$, $H_1 = \Gamma_1/\Gamma$, $H_2 = \Gamma_2/\Gamma$. The surfaces M_1 and M_2 are isometric if and only if Γ_1 and Γ_2 are isometric in $\text{Isom}(\mathbb{H})$. By the previous exercise, we say that M_1 and M_2 are isospectral if they have the same heat trace.

(3) Show that

$$H_t^{M_0}(z, w) = \sum_{\gamma \in \Gamma_0} h_t(z, \gamma w) = \sum_{g \in G} H_t^M(z, gw).$$

and

$$\mathrm{Tr}(H_t^{M_0}) = \frac{1}{|G|} \sum_{g \in G} \int_M H_t^M(z, gz) d\mu(z) = \sum_{[g]} \frac{|[g]|}{|G|} \int_M H_t^M(z, gz) d\mu(z),$$

where the sum is over conjugacy classes of g in G .

(4) (Sunada's criterium for isospectrality [1]) Let $i = 1, 2$. Show that if $H_1, H_2 \in H_i$ are conjugate in G then

$$\int_M H_t^M(z, h_1 z) d\mu(z) = \int_M H_t^M(z, h_2 z) d\mu(z)$$

hence

$$\mathrm{Tr}(H_t^{M_i}) = \sum_{[g]} \frac{|[g] \cap H_i|}{|H_i|} \int_M H_t^M(z, gz) d\mu(z).$$

Conclude that M_1 and M_2 are isospectral if

$$|[g] \cap H_1| = |[g] \cap H_2|$$

for every $g \in G$. If this holds we say that H_1 and H_2 are *almost conjugate* in G . Check that if H_1 and H_2 are conjugate in G then they are almost conjugate.

(5) Show that $H_1 = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$, $H_2 = H_1^T$ are almost conjugate but not conjugate in $G = \mathrm{SL}_3(\mathbb{Z}/2\mathbb{Z})$.

REFERENCES

- [1] T. Sunada, *Riemannian coverings and isospectral manifolds*. Ann. Math. 121 (1985).