EXERCISES 1
ISOMETRIES AND FUNDAMENTAL DOMAINS

(1) Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be an element of $\mathrm{PSL}_2(\mathbb{R})$ different from the identity. Show that
- γ has 1 fixed point in $\mathbb{H}$ iff $|a+d| < 2$;
- γ has 1 fixed point in $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ iff $|a+d| = 2$;
- γ has 2 distinct fixed points in $\hat{\mathbb{R}}$ iff $|a+d| > 2$.

We say that γ is elliptic/parabolic/hyperbolic (respectively).

Show that γ is conjugate (in G) to some k_θ if γ elliptic, n_x if γ is parabolic, a_y if γ is hyperbolic.

(2) Let Γ be a Fuchsian group and $M = \Gamma\backslash\mathbb{H}$ a hyperbolic surface. Show that there is a one-to-one correspondence between the set of conjugacy classes (in Γ) of hyperbolic elements in Γ and closed geodesics on M.

(3) In this exercise you will show that every discrete subgroup $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ that acts properly discontinuously on $\mathbb{H}$ admits a fundamental domain. For this fix $z_0 \in \mathbb{H}$ (such that $\mathrm{Stab}_\Gamma(z_0) = \{e\}$) and consider the *Dirichlet domain*

$$\mathcal{D} = \bigcap_{\substack{\gamma\in\Gamma \\ \gamma\neq e}} \{z \in \mathbb{H} : d_{\mathbb{H}}(z, z_0) < d_{\mathbb{H}}(z, \gamma z_0)\}.$$

(a) Sketch $\mathcal{D}$.
(b) Prove that $\mathbb{H} = \cup_{\gamma\in\Gamma}\, \gamma\overline{\mathcal{D}}$ and $\gamma\mathcal{D} \cap \mathcal{D} = \emptyset$ if $\gamma \neq e$.

(4) Suppose that $\mathcal{F}$ is a fundamental domain for Γ and let Γ^* be the group generated by the set

$$S = \{\gamma \in \Gamma \setminus \{e\} : \gamma\overline{\mathcal{F}} \cap \overline{\mathcal{F}} \neq \emptyset\}.$$

Show that $\Gamma = \Gamma^*$.

(5) Consider a hyperbolic triangle $\mathcal{T}_0$ with a vertex at $i\infty$ (with corresponding inner angle 0) and the two other vertices on the unit circle $S^1 \cap \mathbb{H}$ (with corresponding inner angles α and β.
(a) Show that

$$\mu(\mathcal{T}_0) = \int_{\mathcal{T}_0} \frac{dxdy}{y^2} = \pi - \alpha - \beta.$$

(b) Deduce that any hyperbolic triangle with inner angles α, β, γ has area $\pi - \alpha - \beta - \gamma$.

(6) Let $\mathcal{T}$ be a hyperbolic triangle with inner angles π/a, π/b, π/c and let Γ^* be the group generated by the reflections across each side. Find sufficient conditions on a, b, c for Γ^* to be discrete. Deduce that $\Gamma = \Gamma^* \cap \mathrm{PSL}_2(\mathbb{R})$ is a Fuchsian group.

The *Poincaré polygon theorem* determines under which conditions a convex hyperbolic polygon with an even number of sides is the fundamental domain of a Fuchsian group.