1. Show that
$$V_{1,1} = 2\int_0^\infty \frac{\ell}{1+e^\ell}\,d\ell = \frac{\pi^2}{6}.$$
Hint: $\dfrac{1}{1+e^\ell} = \dfrac{e^{-\ell}}{1+e^{-\ell}} = \sum_{n=1}^{\infty}(-1)^{(n-1)}e^{-n\ell}$.

2. For $S_{1,1}$, the generalized McShane identity in this case is in the form:
$$\sum_\gamma \mathcal{D}(L, \ell_\gamma(X), \ell_\gamma(X)) = L$$
where
$$\mathcal{D}(x,y,z) = 2\log\left(\frac{e^{x/2}+e^{(y+z)/2}}{e^{-x/2}+e^{(y+z)/2}}\right) \qquad \text{and} \qquad \frac{\partial}{\partial L}\mathcal{D}(L,x,x) = \frac{1}{1+e^{x-L/2}} + \frac{1}{1+e^{x+L/2}}.$$
Use this to prove that
$$V_{1,1}(L) = \frac{L^2}{24} + \frac{\pi^2}{6}.$$
Here are some steps:
   (a) Argue, as in the case where $L = 0$, that
   $$L \cdot V_{1.1}(L) = \int_0^\infty x \cdot \mathcal{D}(L,x,x)\,dx.$$
   (b) Show that
   $$\frac{\partial}{\partial L} L \cdot V_{1.1}(L) = \frac{L^2}{8} + \frac{\pi^2}{6}.$$
   (c) Use the fact that $V_{1,1} = \frac{\pi^2}{6}$ to finish the proof.

3. For $X \in M_{1,1}$ and $L > 0$, let $N(X, L)$ be the number of closed curves in X of length at most L. Calculate the average value of $N(X, L)$ over $M_{1,1}$:
$$\frac{1}{V_{1,1}}\int_{M_{1,1}} N(X,L)\,d_{\mathrm{Vol}}(X).$$

4. We say two pants decompositions P and P' differ by an elementary move if
$$P = \{\alpha_1, \ldots, \alpha_i, \ldots, \alpha_{3g-3}\}, \qquad P' = \{\alpha_1, \ldots, \alpha_i', \ldots, \alpha_{3g-3}\}, \qquad i(\alpha_i, \alpha_i') \le 2.$$
Show that any pants decomposition can be transformed to any other pants decomposition using a sequence of elementary moves. Take as a black box the fact that the curve graph $\mathcal{C}(S)$ is connected. The curve graph is a graph whose vertices are curves on S and edges are pairs of disjoint curves.