ISM DISCOVERY SCHOOL AT GAULT
DESCRIPTION AND PLAN OF TALKS

1. Summary

The *relative Langlands program* was born out of the methods used to study automorphic L-functions by representing them by various integrals of automorphic forms. The work of Jacquet, D. Prasad, and others, highlighted the connections between Langlands functoriality and the problem of *distinction* – or, harmonic analysis on certain (almost) homogeneous G-spaces X, such as spherical varieties. The conjectures of Gan–Gross–Prasad and Ichino–Ikeda, based on work of Waldspurger and many others, revealed a pattern that relates *global* integrals of automorphic forms to *local* harmonic analysis. The work of Gaitsgory–Nadler and Sakellaridis–Venkatesh allowed the formulation of a general program, based on the *dual group* of a spherical variety.

The goal of this workshop will be to introduce the more recent work of Ben-Zvi–Sakellaridis–Venkatesh, which takes a step further, introducing a *categorical* version of the relative Langlands program. In it, the "period integrals" of automorphic forms are viewed as "global quantizations" of a Hamiltonian G-space M, and there is a "dual" Hamiltonian $\check{G}$-space $\check{M}$ whose quantization corresponds to an L-function. This is in line with ideas of Kapustin–Witten and, especially, Gaiotto, seeking a correspondence of "boundary conditions" in the quantum-field-theory-interpretation of the geometric Langlands program. The workshop will mostly focus on the analogous *local* conjecture: We will discuss "relative Satake transforms" for spherical varieties, and the relation of associated Plancherel densities to local L-functions, and formulate a categorical version of this relation. This is modelled on the *derived geometric Satake equivalence* due to Ginzburg–Drinfeld–Bezrukavnikov–Finkelberg, which we view as a categorification of Macdonald's formula for zonal spherical functions and the unramified Plancherel measure.

2. Organization of lectures

Sakellaridis will be responsible for explaining the second paragraph of the summary above. There will be other talks explaining the first paragraph and cultivating an appreciation of the number-theoretic background (especially, the Ichino–Ikeda conjecture, but also classical constructions such as Poincaré, Eisenstein, and theta series); and talks providing the geometric and categorical background for this discussion.

3. Schedule of talks, with some references

(1) Wednesday pm: Poincaré, Eisenstein, and theta series. "Periods" as integrals against theta series. *(Mathilde Gerbelli-Gauthier)*
Reference(s): [Sak12]

(2) Wednesday pm: (Derived categories of) constructible sheaves and perverse sheaves on schemes. *(Antoine Labelle)*

Reference(s): [Ach21], [Wil], [AGK$^+$20, Appendix E], [BZSV23, Appendix B]

(3) Thursday am: Periods, local distinction, the Gan–Gross–Prasad and Ichino–Ikeda conjectures. *(Jialiang Zou)*

Reference(s): [GGP12], [II10], [SV17, Chapters 16–17]

(4) Thursday am: Constructible sheaves on stacks; Koszul duality. "$(1-a)(1+a+a^2+\cdots)=1$." *(Taeuk Nam)*

Reference(s): [BL94], [BGS96], [AGK$^+$20, Appendix F], [BZSV23, Appendices A–B]

(5) Thursday pm: Satake isomorphism and the Macdonald formula. (Yiannis)

Reference(s): [Gro98], [Zhu20], [Cas80], [Aut23, Tag 00IK].

(6) Thursday pm: The derived Satake equivalence. *(Calder Morton-Ferguson)*

Reference(s): [BF08], [Gin], [Zhu17]

(7) Friday am: "Relative" Satake isomorphisms and the unramified Plancherel formula for spherical varieties. (Yiannis)

Reference(s): [Sak18], [Sak13]

(8) Friday am: The L-function associated to a spherical variety. Duality of Hamiltonian spaces. *(Eric Chen)*

Reference(s): [Sak13], [BZSV23, Chapter 4]

(9) Friday pm: Conjectural "relative" derived equivalence. (Toan Pham)

Reference(s): [BZSV23, Chapters 7–9]

(10) Friday pm: Further discussion of the relative derived equivalence. (Yiannis)

Reference(s): [BZSV23, Chapters 7–9]

(11) Saturday am: Generalization to singular spaces: the IC function, and various examples. (Yiannis)

Reference(s): [BFGM02, BNS16, SW22]

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